Sobolev-Orlicz inequalities for Dirichlet forms & their applications to spectral theory

by

Ali Ben Amor at Gafsa

(X,δ,m) a loc. comp. separable metric-measure space, $\mathcal{E}$ a Dirichlet form on $L^2(X,m)$, μ a positive Radon smooth measure on X, Φ an N-function, $L^\Phi(X,\mu)$ the Orlicz space rel. to Φ and

$$\text{(SOI)} := \|f^2\|_{L^\Phi(X,\mu)} \leq C\mathcal{E}(f,f),\ \forall\, f \in D(\mathcal{E}).$$

Goals:

1) Give NSC for the validity of (SOI).
2) Use (SOI) to get Nash-type inequality and Ultracontractivity of $e^{-tH},\ t > 0$ (for $\mu = m$).
3) Use (SOI) to get informations on the spectrum of the trace of $\mathcal{E}$ on the support of μ.
4) Applications: Traces of $(-\Delta)^s$ and DF on Besov spaces (UC and spectra).

Two main objects: *Orlicz spaces* and *Dirichlet forms*.

$\Phi : [0,\infty) \to [0,\infty)$ is a *Young's function* if it is convex, $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$.

If $\Phi(x) = 0 \Leftrightarrow x = 0$, $\lim_{x\to 0} \Phi(x)/x = 0$ and $\lim_{x\to\infty} \Phi(x)/x = \infty$, then Φ is an N-function.

To $\Phi \leftrightarrow \Psi$:

$$\Psi(x) = \sup\Big\{yx - \Phi(y) : y \geq 0\Big\},\ x \geq 0.$$

Ψ *complementary* func. of Φ. Φ N-func. then Ψ also is.

Orlicz space $L^{\Phi}(\mu) := L^{\Phi}(X,\mu)$: the space of μ-meas. (eq. classes) of functions f s.t.

$$\|f\|_{L^{\Phi}(\mu)} := \sup\Big\{\Big|\int_X fg\,d\mu\Big| : \int_X \Psi(|g|)\,d\mu \leq 1\Big\} < \infty$$

$\Big(L^{\Phi}(\mu), \|\cdot\|_{L^{\Phi}(\mu)}\Big)$ is a Banach space.

Set

$$\|f\|_{(\Phi)} := \inf\Big\{\lambda > 0 : \int_X \Phi(|f|/\lambda)\, d\mu \leq 1\Big\}.$$

Then

$$\|f\|_{(\Phi)} \leq \|f\|_{L^{\Phi}(\mu)} \leq 2\|f\|_{(\Phi)}, \ \forall f \in L^{\Phi}(\mu).$$

Example: $1 < p < \infty$, q s.t. $p^{-1} + q^{-1} = 1$.

$$\Phi(t) = \frac{t^p}{p},\ t \geq 0, \text{then } \Psi(t) = \frac{t^q}{q},\ t \geq 0$$

and $\|f\|_{L^{\Phi}(\mu)} = q^{1/q}\|f\|_p$.

Dirichlet forms: A DF is a densely defined positive quad. form on $D(\mathcal{E}) \subset L^2(X,m)$ s.t.: $\forall\ 0 \leq a \leq b$

$$f \in D(\mathcal{E}) \Rightarrow\ f_{a,b} := \Big(\min(f,b) - a\Big)_+ \in D(\mathcal{E})$$

and

$$\mathcal{E}[f_{a,b}] \leq \mathcal{E}[f] := \mathcal{E}(f,f).$$

Regularity: $D(\mathcal{E}) \cap C_c(X)$ is dense both in $D(\mathcal{E})$ and in $C_c(X)$ w.r.t. the relative norms.
Transience: $\exists g \in L^1$, $g > 0$-a.e. such that

$$\Big| \int_X f g \, dm \Big| \leq C\sqrt{\mathcal{E}[f]}, \ \forall\, f \in D(\mathcal{E}).$$

$$\mathcal{E} \leftrightarrow H \leftrightarrow T_t := e^{-tH}, \ t > 0.$$

Examples: 1) $\mathbb{R}^n$ the n-dim. Euc. space with Lebesgue meas. dx, $W^{1,2}(\mathbb{R}^n)$ the first order Sob. space on $\mathbb{R}^n$.

$$\mathcal{E}[f] := \int_{\mathbb{R}^n} |\nabla f|^2 \, dx, f \in W^{1,2}(\mathbb{R}^n)$$

is a regular DF in $L^2(\mathbb{R}^n, dx)$.
2) Define

$$\Gamma(f)(x) := \limsup_{\delta(x,y)\to 0} \frac{|f(x) - f(y)|}{\delta(x,y)}, \ f \in \mathrm{Lip(X)},$$

$$\mathcal{E}[f] := \int_X \Gamma^2(f)(x) \, dm, \ f \in \mathrm{Lip(X)}.$$

$\mathcal{E}$ is a regular pre-DF in $L^2(X, m)$.

From now on $\mathcal{E}$ is transient.
The *Capacitity*: $\forall\,\Omega\subset X$, open,

$$\mathrm{Cap}(\Omega) := \inf\{\mathcal{E}[f] : f\in D(\mathcal{E}),\ f\geq 1 \text{ a.e. on } \Omega\},$$

In the sequel, we assume that all measures do not charge sets having zero capacity.

Observation: $\|f^2\|_{L^\Phi}\sim\|f\|^2_{L^{\Phi_2}}$, where $\Phi_2(t)=\Phi(t^2)$.

$$(\mathrm{SOI}) := \|f^2\|_{L^\Phi(X,\mu)}\leq C\mathcal{E}[f],\ \forall\, f\in D(\mathcal{E}).$$

$\Leftrightarrow$ boundedness

$$I_\mu : D(\mathcal{E})\to L^{\Phi_2}(X,\mu),\ f\mapsto f.$$

Hereafter, Ψ_2 is the complementary function of Φ_2.

Popular formulations of SOI

$$\text{(SOI)} := \|f^2\|_{L^{\Phi}(X,\mu)} \le C\mathcal{E}[f], \ \forall f \in D(\mathcal{E}).$$

a) $\Phi(t) = t^{\alpha}, \ \alpha > 1$, (SOI) is:

$$\text{(SI)} := \Big(\int_X f^{2\alpha}\, d\mu \Big)^{1/\alpha} \le C\mathcal{E}[f], \ \forall f \in D(\mathcal{E}).$$

b) $\Phi(t) = t \log(1+t), \ t \ge 0$ and $\mu = m$, (SOI)$\Rightarrow$ the log.Sob.I.:$\forall f \in D(\mathcal{E})$,

$$\text{(LSI)} := \int_X f^2 \log \Big(\frac{f^2}{\int_X f^2\, dm} \Big)\, dm \le C\mathcal{E}[f],$$

SOI: Necessary and sufficient conditions

Theorem 1. Φ *an N-function. Then the following assertions are equivalent:*

i) $\exists C_1$ *s.t.*

$$\|f^2\|_{L^{\Phi}(\mu)} \le C_1 \mathcal{E}[f], \ \forall f \in D(\mathcal{E}). \tag{1}$$

ii) $\exists C_2$ *s. t.* $\forall K \subset X$, *compact*

$$\text{(CI)} := \mu(K)\Psi^{-1}(1/\mu(K)) \le C_2 \text{Cap}(K).$$

iii) $\forall f \in L^{\Psi_2}(\mu)$, $f\mu$ *has finite energy integral and*

$$K^{\mu} := L^{\Psi_2}(\mu) \to L^{\Phi_2}(\mu), \ f \mapsto U(f\mu),$$

is bounded.

Furthermore, $C_2 \le C_1 = 4\|K^{\mu}\| \le 4C_2$.

For $\Phi(t) = t^{\alpha}, \ \alpha > 1$, the (CI) reads

$$(\mu(K))^{1/\alpha} \le C \text{Cap}(K).$$

For the gradient DF in $L^2(\mathbb{R}^n, dx), \ n > 2$, Mazy'a (70'):

$$(CI) \Leftrightarrow \sup_{x, 0<r\le 1} \frac{(\mu(B_r(x)))^{1/\alpha}}{r^{n-2}} < \infty.$$

Consequences: a) *Nash-type* ineq.: Set

$$\Lambda(s) := \frac{1}{s\Phi^{-1}(1/s)}, \ s > 0.$$

Theorem 2. *If (SOI) holds true with best constant* κ. *Then* $\forall f \in D(\mathcal{E})$, $\epsilon \in (0,1)$

$$(NI) := \mathcal{E}[f] \geq \kappa^{-1}(1-\epsilon)\left(\int_X f^2\, d\mu\right)\Lambda\Big(\frac{2\|f\|^2_{L^1(\mu)}}{\epsilon \int_X f^2\, d\mu}\Big).$$

$\Phi(t) = t^{n/n-2}$, $n > 2$, $\Lambda(s) = cs^{-2/n}$. If $\mu = m$, then (SOI) is (SI) and (NI) reads:

$$\Big(\int_X f^2\, dm\Big)^{(1+2/n)} \leq C\|f\|^{4/n}_{L^1}\mathcal{E}[f], \ \forall f \in D(\mathcal{E}).$$

b) *Ultracontractivity*: An N-function Φ is said to be *admissible* if

$$\int_0^\alpha \Big(s\Lambda(s)\Big)^{-1} ds < \infty \text{ for some } \alpha > 0.$$

For such a function, set

$$t := 8\kappa \int_0^{\gamma(t)} \frac{1}{s\Lambda(s)}\, ds,$$

and for every $\epsilon \in (0,1)$

$$\beta_\epsilon(t) := \frac{2}{\epsilon\gamma(2(1-\epsilon)t)},\ \ t > 0.$$

Theorem 3. *Assume (SOI) with an admissible N-function* Φ *and* $\mu = m$*. Then for every* $t > 0$*,* T_t *is ultracontractive and*

$$\|T_t\|_{L^1,L^\infty} \le \beta_\epsilon(t/2),\ \ \forall \epsilon \in (0,1). \tag{2}$$

$\Rightarrow \forall t > 0$*,* T_t *is an integral operator, with kernel* $p_t(x,y)$ *s.t.*

$$\operatorname*{ess\,sup}_{x,y\in X} p_t(x,y) \le \beta_\epsilon(t/2),\ \ \forall t > 0. \tag{3}$$

Application: Lower bounds for eigenvalues of traces of DF

$\mathcal{F}_e$ the extended domain of $\mathcal{E}$, μ a measure with support F.
The *trace* of $\mathcal{E}$ on $L^2(F,\mu)$ is

$$D(\check{\mathcal{E}}) := \{f \in L^2(F,\mu)\colon f = u - \mu \text{ a.e. on } F \\ \text{for some } u \in \mathcal{F}_e\},$$

$$\check{\mathcal{E}}[f] = \inf\{\mathcal{E}[u] : u \in \mathcal{F}_e,\ u = f - \mu \text{ a.e. on } F\}.$$

It is a transient DF in $L^2(F,\mu)$. Set $\check{H} \leftrightarrow \check{\mathcal{E}}$.
If (SOI) holds true for $D(\mathcal{E})$ with an admissible N-function
$\Rightarrow$ it holds also true for $D(\check{\mathcal{E}})$. If μ is finite, by Theorem 3, $\check{H}$ has discrete spectrum and the first eigenvalue satisfies the Faber-Krahn-type inequality

$$\check{\lambda}_1 \geq C(\mu(F)\Phi^{-1}(1/\mu(F)))^{-1} = C\Lambda(\mu(F)). \quad (4)$$

$(\check{\lambda}_k)_{k\geq 1}$ the eigenvalues of $\check{H}$, ordered in an $\uparrow$ way.

Theorem 4. *Under the above assumptions,*

$$\check{\lambda}_k \geq \frac{1-\epsilon}{16\kappa_\mu}\Lambda\Big(\gamma(2\gamma^{-1}(\frac{2\mu(F)}{k\epsilon}))\Big), \forall\, 0 < \epsilon < 1. \quad (5)$$

Examples: a) s-stable processes

Let $0 < s \leq 1$ and $\mathcal{E}^{(s)}$:

$$D(\mathcal{E}^{(s)}) = \Big\{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\hat{f}(x)|^2 |x|^{2s}\, dx < \infty\Big\},$$

$$\mathcal{E}^{(s)}(f,g) = \int_{\mathbb{R}^n} \hat{f}(x)\overline{\hat{g}(x)}|x|^{2s}\, dx.$$

$\mathcal{E}^{(s)}$ is a reg. DF, $D(\mathcal{E}^{(s)}) = L^{s,2}(\mathbb{R}^n)$.
μ a *semi d-measure*:

$\exists\, 0 < d \leq n$, $C_1 \geq 0$, $0 < C_2 < 1$ and $b \in \mathbb{R}$ s.t.

$$\mu(B_x(r)) \leq C_1 r^d \Big|\log(C_2 r)\Big|^b, \ \forall\, r \in (0,1], x \in F.$$

For $n > 2s$, $n - 2s < d \leq n$ and $2 < p < \frac{2d}{n-2s}$ ($p \leq \frac{2d}{n-2s}$ if $b = 0$), then

$$\Big(\int_X |f|^p\, d\mu\Big)^{2/p} \leq C\mathcal{E}^{(s)}[f], \ \forall\, f \in D(\mathcal{E}^{(s)}).$$

$\Rightarrow$ if $(\check{\mathcal{E}^{(s)}})$ is the trace of $\mathcal{E}^{(s)}$ on $L^2(F,\mu)$, then

$$\check{p}_t \leq C\frac{1}{t^{p/p-2}}\ \mu \times \mu \text{ a.e., } \forall\, t > 0,$$

If further μ is finite, then $(\check{\mathcal{E}^{(s)}})$ has discrete spectrum and

$$\check{\lambda}_k \geq Ck^{\frac{p-2}{p}},\ \forall\, k \geq 1.$$

b) A DF on a Besov space:

μ a *d-measure*:

$$\mu(B_x(r)) \sim r^d,\ \forall\, r \in (0,1],\, x \in F,$$

$n = 2s,\ \alpha := s + \frac{d}{2} = \frac{n+d}{2}$, and $\Phi(t) := e^t - 1$.
Λ is then given by

$$\Lambda(s) = \frac{1}{s \log(1/s + 1)}.$$

$\mathcal{B}$ defined by

$$\mathcal{B}[f] := \int_{F\times F\setminus\Delta} \frac{(f(x) - f(y))^2}{|x-y|^{d+2\alpha}}\, d\mu(x)\, d\mu(y),$$

$$D(\mathcal{B}) := \{f \in L^2(F,\mu) : \mathcal{B}[f] < \infty\}.$$

Fukushima (03): $\mathcal{B}$ is a reg. DF of *jump type* on $L^2(F,\mu)$ and $D(\mathcal{B})$ is the Besov space $B_\alpha^{2,2}(F)$.

Proposition 1. *i)*

$$\|f^2\|_{L^\Phi(\mu)} \leq C\mathcal{B}_1[f],\ \forall f \in B_\alpha^{2,2}(F),$$

$\Rightarrow$ the semigroup of the operator related to $\mathcal{B}$ has an a.c. (w.r.t. to μ) kernel p_t^μ.

ii) If μ is finite, then $\mathcal{B}$ has discrete spectrum and $\forall k \geq 1$

$$\lambda_k \geq C\frac{(1-\epsilon)}{2\mu(F)}\frac{k\epsilon}{\log(\frac{k\epsilon}{2\mu(F)}+1)},\ \forall\, 0 < \epsilon < 1.$$