Improved Schwarz Methods for an Elliptic Problem in a Nonconvex Domain with Discontinuous Coefficients

CHNITI Chokri

Mâitre assistant à l'IPEIN

ISMAI Kairouan

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Overview

- Domain decomposition Methods
- Additive Schwarz Method
- Lions Algorithm
- Our algorithm
  - Case of regular interface
  - Case of corner singularities
- Conclusion

Domain Decomposition Methods

Global problem

The idea: Let $\mathcal{L}$ be an elliptic operator and Ω is a domain.
We consider $\mathcal{L}u = f$ in Ω, $\quad +B.C$ on $\partial\Omega$.

Subproblem

If Ω is "Large", it's can be decomposed into subdomains, $\overline{\Omega} = \cup_{i=1}^{i=N}\overline{\Omega_i}$ where Ω_i is an open subdomain of Ω. We solve in iterative way

$$\begin{cases} \mathcal{L}u_i^{n+1} & = f \quad \text{in} \quad \Omega_i \\ \mathcal{S}_{ij}u_i^{n+1} & = \mathcal{S}_{ij}u_j^n \quad \text{on} \quad \partial\Omega_i \cap \overline{\Omega_j} (i \neq j) \\ +B.C & \text{on } \partial\Omega_i \cap \partial\Omega \end{cases}$$

for $i = 1, \ldots, N$.

$\Rightarrow$ Importance of the choice of the transmission conditions.

The additive Schwarz Method (H.A. Schwarz, 1870)

For $\eta \geq 0$, $\begin{cases} (\eta - \Delta)u & = f \quad \text{in } \Omega \\ u & = 0 \text{ on } \partial\Omega \end{cases}$

$(u_1^n, u_2^n) \longrightarrow (u_1^{n+1}, u_2^{n+1})$ where

$$\begin{cases} (\eta - \Delta)u_1^{n+1} & = f \quad \text{in } \Omega_1 \\ u_1^{n+1} & = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{n+1} & = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2} \end{cases} \quad \begin{cases} (\eta - \Delta)u_2^{n+1} & = f \quad \text{in } \Omega_2 \\ u_2^{n+1} & = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ u_2^{n+1} & = u_1^n \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1} \end{cases}$$

⊕ **A simple and convergent algorithm.**

⊖ **Slow convergence, we need a domain decomposition with overlapping.**

Lions Algorithm (P.L. Lions, 1988)

$$\begin{cases} (\eta - \Delta)u_1^{n+1} & = f \quad \text{in } \Omega_1, \\ u_1^{n+1} & = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ (\frac{\partial}{\partial n_1} + \alpha)(u_1^{n+1}) & = (-\frac{\partial}{\partial n_2} + \alpha)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}, \end{cases}$$

$$\begin{cases} (\eta - \Delta)u_2^{n+1} & = f \quad \text{in } \Omega_2, \\ u_2^{n+1} & = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \\ (\frac{\partial}{\partial n_2} + \alpha)(u_2^{n+1}) & = (-\frac{\partial}{\partial n_1} + \alpha)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}, \end{cases}$$

with $\alpha > 0$.

$\oplus$ **Using Robin interface conditions, the overlapping is not anymore necessary**.

Our aim

We consider an elliptic operators with discontinuous coefficients.
We propose

- the form of the transmission conditions in the case of regular interface and the case of corner singularities,
- the strategy of optimization in each case,
- the optimal form of the transmission conditions.

Case of regular interface

- Let $\Omega = \mathbb{R}^2 = \Omega_1 \cup \Omega_2$, $\Omega_1 = (-\infty, 0) \times \mathbb{R}$ and $\Omega_2 = (0, +\infty) \times \mathbb{R}$.
- We consider the problem

$$\left\{ \begin{array}{rcll} -\nabla.(\nu(x)\nabla u)u & = & f \text{ for} & x \in \Omega \\ |u| & < & +\infty \text{ as} & x \to \infty \end{array} \right.$$

where f is the right-hand side and

$$\nu(x) = \left\{ \begin{array}{l} \nu_1 \text{ in } \Omega_1 \\ \nu_2 \text{ in } \Omega_2 \end{array} \right.$$

- At the interface $x = 0$:

$$u(0^+, y) = u(0^-, y) \text{ and } \nu_1 \frac{\partial u}{\partial x}(0^+, y) = \nu_2 \frac{\partial u}{\partial x}(0^-, y), \quad y \in \mathbb{R}$$

Schwarz Algorithm

If (u_1^n, u_2^n) are known, the step $n+1$ is determined by solving:

$$\left\{\begin{array}{rl} -\nu_1 \Delta u_1^{n+1} & = 0 \quad \text{in} \quad \Omega_1 \\ (\nu_1 \frac{\partial}{\partial n_1} + \mathcal{S}_2) u_1^{n+1}(0,y) & = (-\nu_2 \frac{\partial}{\partial n_2} + \mathcal{S}_2) u_2^n(0,y), \quad y \in \mathbb{R} \\ |u_1^{n+1}| & < \infty \end{array}\right.$$

$$\left\{\begin{array}{rl} -\nu_2 \Delta u_2^{n+1} & = 0 \quad \text{in} \quad \Omega_2 \\ (\nu_2 \frac{\partial}{\partial n_2} + \mathcal{S}_1) u_1^{n+1}(0,y) & = (-\nu_1 \frac{\partial}{\partial n_1} + \mathcal{S}_1) u_1^n(0,y), \quad y \in \mathbb{R} \\ |u_2^{n+1}| & < \infty \end{array}\right.$$

where $\mathcal{S}_j$, $j = 1, 2$, are linear operators acting in the y direction only.

A natural question

How do we choose the transmission conditions in the Schwarz method to get fast convergence of the iteration?
$\Rightarrow$ Use the same strategy introduced first by [C.Japhet] for the advection-diffusion equation.

Solution

- Good choices of the transmission conditions optimize the performance of the Schwarz iteration.
- Fix a certain class of local transmission conditions and optimize the convergence factor of the iteration over this class.
- Need an explicit expression for the convergence factor ρ.
- For the reaction-diffusion problems, the convergence can be fully analyzed and using a Fourier transform a convergence factor $\rho(k)$ can be obtained.

Convergence factor

- Denote $e_i^n = u_i^n - u$.
- Solve the problem with Fourier transform applied to y. The convergence rate is defined as:

$$\rho(k) = \frac{\widehat{e_i^{n+2}}(0,k)}{\widehat{e_i^n}(0,k)}$$

$$\rho(k) = \left| \frac{\sigma_2(k) - \nu_2 k}{\sigma_2(k) + \nu_1 k} \quad \frac{\sigma_1(k) - \nu_1 k}{\sigma_1(k) + \nu_2 k} \right|$$

where $\sigma_j(k)$ are the Fourier symbols of the operators $\mathcal{S}_j, j = 1, 2$.

Example

If we consider

$$\mathcal{S}_j = \nu_j \left(\beta - \frac{\partial}{\partial \tau}\left(\frac{\alpha}{2}\frac{\partial}{\partial \tau}\right)\right), \quad \sigma_j(k) = \nu_i\left(\beta + \frac{\alpha}{2}k^2\right)$$

the convergence factor is

$$\rho(k;\alpha,\beta) = \frac{(\beta + \frac{\alpha}{2}k^2 - k)^2}{(\beta + \frac{\alpha}{2}k^2 + \mu k)(\beta + \frac{\alpha}{2}k^2 + \frac{k}{\mu})}, \qquad \mu = \frac{\nu_1}{\nu_2}, \qquad 0 < k_1 < k_2\,.$$

Optimization

Theorem

The min-max problem $\min_{\alpha>0,\beta\geq 0}\max_{0<k_1\leq k\leq k_2}|\rho(k;\alpha,\beta)|$ *has a unique solution given by:*

- $$k^{\star}=\sqrt{\frac{2\beta}{\alpha}}$$
- $$\beta_{opt}=\frac{(k_1k_2)^{\frac{3}{4}}}{\sqrt{2(k_1+k_2)}}$$
- $$\alpha_{opt}=\frac{2(k_1k_2)^{-\frac{1}{4}}}{\sqrt{2(k_1+k_2)}}$$

- k_1 is the smallest frequency relevant to the subdomain,
- k_2 is the largest frequency supported by the numerical grid.

Question

Can we use the interface conditions of the regular case in the vicinity of the corner?

$\Rightarrow$ The ingredient of our study is the Mellin transform

$$\widehat{u}(z) = \mathcal{M}(u)(z) = \int_0^{\infty} r^{iz} u(r) \frac{dr}{r}$$

Solution

Polar coordinates + Mellin transform $\Rightarrow \begin{cases} \left(\frac{\partial^2}{\partial\theta^2} - z^2\right)\widehat{e_1^{n+1}}(z,\theta) = 0 \\ \widehat{e_1^{n+1}}(z,\theta_+) = \widehat{e_2^n}(z,\theta_+) \\ \widehat{e_1^{n+1}}(z,\theta_-) = \widehat{e_2^n}(z,\theta_-) \end{cases}$

$\Rightarrow$ Slow convergence and overlapping is necessary.

New interface conditions near the corner

Our interface condition:

$$\mathcal{S}_{ij} = \nu_i \frac{\partial}{\partial n_i} + \nu_j \left(\frac{\beta_i}{r} - \frac{1}{2} \frac{\partial}{\partial r} (\alpha_i r \frac{\partial}{\partial r}) \right)$$

where $\frac{\partial}{\partial n_i}$ is the outward normal derivative on $\partial\Omega_i$, r is the distance to corner and $\alpha_i, \beta_i \in \mathbb{R}_+$.

$\Rightarrow$ All the operators have the same homogeneity degree -1.

Subproblem in Ω_1

Following Kondratiev's Theory: Polar cordinates (r,θ) and principal part $(r \longrightarrow 0)$

$$\left\{\begin{array}{rl} \left((r\partial_r)^2+\partial_\theta^2\right) e_1^{n+1}(r,\theta) & = 0 \\ \left(\nu_1\partial_\theta-\nu_2\beta_1+\nu_2\frac{\alpha_1}{2}(r\partial_r)^2\right) e_1^{n+1}(r,\theta_-) & = g^n(r) \\ e_1^{n+1}(r,\theta_+) & = 0 \end{array}\right. \tag{1}$$

where

$$g^n(r) = \left(\nu_2\partial_\theta-\nu_2\beta_1+\nu_2\frac{\alpha_1}{2}(r\partial_r)^2\right) e_2^n(r,\theta_-).$$

Polar coordinates, principal part and Mellin tronsform

$$\left\{\begin{array}{rl} \left(\partial_\theta^2-z^2\right) \widehat{e_1^{n+1}}(z,\theta) & = 0 \\ \left(\nu_1\partial_\theta-\nu_2\beta_1-\nu_2\frac{\alpha_1}{2}z^2\right) \widehat{e_1^{n+1}}(z,\theta_-) & = \widehat{g^n}(z) \\ \widehat{e_1^{n+1}}(z,\theta_+) & = 0 \end{array}\right.$$

Solution

$$\widehat{e_1^{n+1}}(\theta) = a(z)e^{z(\theta-\theta_-)} + b(z)e^{-z(\theta-\theta_+)}$$
$$a(z) = \mathcal{R}(z)\widehat{g^n}(z),\ b(z) = -a(z)e^{z(\theta_+-\theta_-)}$$
$$\mathcal{R}(z) = \left[\left(\nu_1 z - \nu_2\beta_1 - \nu_2\frac{\alpha_1}{2}z^2\right) + \left(\nu_1 z + \nu_2\beta_1 + \nu_2\frac{\alpha_1}{2}z^2\right)e^{2z(\theta_+-\theta_-)}\right]^{-1}$$

Proposition

The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are $z = it_k$ with

$$\tan\left(t_k(\theta_+ - \theta_-)\right) = \frac{2t_k\mu}{\alpha_1 t_k^2 - 2\beta_1}, \quad \mu = \frac{\nu_1}{\nu_2}, \quad 0 < t_1 < t_2 < \ldots < t_n < \ldots$$

Remarks

- The strength of the singularity is related to the angle of the corner. The subproblems have smaller angles than the original problem. For that reason, the decomposition of a polygonal domain by an interface arriving at a corner reduces the angle of the corner, which corresponds to a greater regularity of the subproblems as compared to the original problem.
- The final aim of our work is to improve the convergence of domain decomposition methods in the neighborhood of corners. A new strategy of optimization of the transmission conditions in the vicinity of the corners will be proposed.

Strategy of optimization near the corner

First strategy

First, check if it is possible to cancel the first artificial pole (nearest to the origin and associated to the subdomain) for a well prepared data: i.e. assume that at the step n the error function $e_j^n = u_j^n - u$, $j = 1, 2$, have the asymptotic type of the global problem, then the error e_j^{n+1} should keep this asymptotic type up to some large enough order as $r \longrightarrow 0$.

Second strategy

If the first strategy has no solution, choose the pairs $(\alpha_i, \beta_i), i = 1, 2$, so that the first artificial pole has the largest possible imaginary part.

Mellin transform

$$\mathcal{M}(u)(z) = \int_0^\infty r^{iz} u(r) \frac{dr}{r}$$

If $u \in \mathbf{C}_0^\infty([0, +\infty))$ and

$$u(r) = cr^{-i\psi} + O(r^\gamma), \gamma > \mathrm{Im}(\alpha)$$

then $\mathcal{M}(u)(z)$ is

- holomorphic in $\mathrm{Im}(z) < \mathrm{Im}(\psi)$,
- meromorphic in $\mathrm{Im}(z) > \mathrm{Im}(\psi)$ and
- has a simple pole of residue c in $z = \psi$.

A symmetric decomposition

If $u \in H^1(\Omega)$ satisfies $(u) \subset \{r \leq 1\}$ then the expansion of u is

$$u(r,\theta) = \sum_{n \in \mathbb{N}^*, \quad \frac{n\pi}{\theta_+ - \theta_0} \leq 1+\gamma} \nu_n E_n(r,\theta) + o\left(r^{\left[\frac{(1+\gamma)(\theta_+ - \theta_0)}{\pi}\right] \frac{\pi}{\theta_+ - \theta_0}} \right), \quad \gamma \in]0,1[\tag{2}$$

where $\nu_n \in \mathbb{R}$,

$$E_n(r,\theta) = \begin{cases} r^{\frac{n\pi}{\theta_+ - \theta_0}} \sin\left(\frac{n\pi}{\theta_+ - \theta_0} (\theta - \theta_+) \right) & \text{if } n \text{ is odd} \\ \frac{1}{\nu} \quad r^{\frac{n\pi}{\theta_+ - \theta_0}} \sin\left(\frac{n\pi}{\theta_+ - \theta_0} (\theta - \theta_+) \right) & \text{if } n \text{ is even} \end{cases}$$

and $\left[\frac{(\gamma+1)(\theta_+ - \theta_0)}{\pi}\right]$ denotes the integer part of the real number $\frac{(\gamma+1)(\theta_+ - \theta_0)}{\pi}$.
The cancellation of the first artificial pole it_1 is reduced to $\beta_1 = \frac{\alpha_1 \pi^2}{2(\theta_+ - \theta_0)^2}$.

A non-symmetric decomposition

Case where $\theta_+ - \theta_- = \frac{\pi}{2}$. The first asymptotic type of the global solution is

$$u(r,\theta) = r^{T_1} w(\theta) + o(r^{T_1})$$

where

$$w(\theta) = \begin{cases} \sin\left(T_1(\theta - \theta_0)\right) \text{ if } \theta_0 \leq \theta \leq \theta_- \\ -\sqrt{\frac{2\nu_2}{\nu_1+\nu_2}} \sin\left(T_1(\theta - \theta_+)\right) \text{ if } \theta_- \leq \theta \leq \theta_+ \end{cases}$$

and $T_1 = 1 - \frac{1}{\pi} \arccos\left(\frac{\nu_1}{\nu_1+\nu_2}\right)$. The cancellation of the first artificial pole it_1 is reduced to

$$\beta_1 = \frac{\alpha_1}{2} T_1^2 - \frac{T_1}{\sqrt{\frac{\nu_2}{\nu_1}(2+\frac{\nu_2}{\nu_1})}}$$

Reaction diffusion problem

We consider an elliptic equation with highly jumping coefficients

$$\begin{cases} -\nabla.(\nu(x)\nabla u) + \eta(x)u &= f \text{ in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \text{ on } \gamma, \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \setminus \gamma, \end{cases}$$

- The poles of $\widehat{e_1^{n+1}}$ have the form $z_k = it_k$, with $0 < t_1 < t_2 < t_3 < \dots$
- The cancellation of the first artificial pole it_1 gives

$$\alpha_1 = \frac{2}{\tau_1} \frac{1}{\tan(\frac{\theta_+ - \theta_- - 2\pi}{2}\tau_1)} \text{ for } \frac{\theta_+ - \theta_-}{\pi} \in]2 - \frac{2}{\tau_1}, 2 - \frac{1}{\tau_1}[$$

where τ_1 denote the first positive solution of

$$\sin^2(\pi t) = \gamma \sin^2(\Theta t)$$

with $\gamma = (\frac{\nu_2 - \nu_1}{\nu_2 + \nu_1})^2, \qquad \Theta = \theta_+ - \theta_- - \pi$,

- $\beta_1 = 0$.

Case where Ω is a disc. Notations:

- $$|u|_{L^\infty(\Omega)} = ||u||_{L^\infty(\Omega)}$$
- $$|u|_1 = \left(\int_\Omega |\nabla u|^2(x)dx \right)^{1/2}$$

In our comparison of numerical methods, we shall use the terminology

- **ICCC:** for the interface conditions with constant coefficients $(\alpha_{opt}, \beta_{opt})$ up to the corner.
- **OCC:** for the new interface condition with optimized coefficients at the corner (α_1, β_1).

Examples

First case

If $\theta_+ - \theta_- = \frac{\pi}{2}$, then

- $$\tau_1(\frac{\pi}{2}) = \frac{1}{\pi}\arccos\left(-\frac{\nu_1^2+\nu_2^2+6\nu_1\nu_2}{2(\nu_1+\nu_2)^2}\right) \in]\frac{2}{3},1[,$$
- $$\alpha_1 = -\frac{2}{\tau_1}\frac{1}{\tan(\frac{3\pi}{4}\tau_1)} > 0, \Rightarrow \text{First strategy will be applied},$$
- if $\nu_1 = 1$, then $\tau_1(\nu_2)$ decreasing with ν_2 to the value $\frac{2}{3}$ and the solution of the interface problem belongs to $H^{1+\frac{2}{3}}(\Omega)$.

Second case

If $\theta_+ - \theta_- = \frac{3\pi}{2}$, then $\tau_1(\frac{3\pi}{2}) = \tau_1(\frac{\pi}{2})$ and
$\alpha_1 = -\frac{2}{\tau_1}\frac{1}{\tan(\frac{\pi}{4}\tau_1)} < 0 \Rightarrow$ Second strategy will be applied.

Number of iterations

ν_2	2	4	6	8	10	10^2	10^3	10^4	10^5	10^6
Iterations **OCC**	12	10	8	8	7	7	3	3	2	1
Iterations **ICCC**	22	14	11	9	8	7	3	3	2	1

Table: Refined mesh around the corner. Number of iterations for different values of ν_2 with $|e_1^n|_1 < 10^{-6}$

Iteration	2	6	8	10	12
OCC	$8.5\ 10^{-3}$	$7.1\ 10^{-5}$	$1.1\ 10^{-5}$	$2.6\ 10^{-6}$	$6.5\ 10^{-7}$
ICCC	$8.5\ 10^{-3}$	$7.5\ 10^{-5}$	$2.7\ 10^{-5}$	$1.4\ 10^{-5}$	$8.2\ 10^{-7}$
$\frac{\lvert e_1^n\rvert_{\infty,\mathbf{ICCC}}}{\lvert e_1^n\rvert_{\infty,\mathbf{OCC}}} \simeq$	1.002	1.07	2.33	5.49	12.67

Table: Case with $\nu_1 = 1, \nu_2 = 2$. Comparaison of the $|e_1^n|_\infty$ near the corner with **OCC** and **ICCC** with respect to the number of iterations.

Plot $\log_{10} |e_1^n|_1$

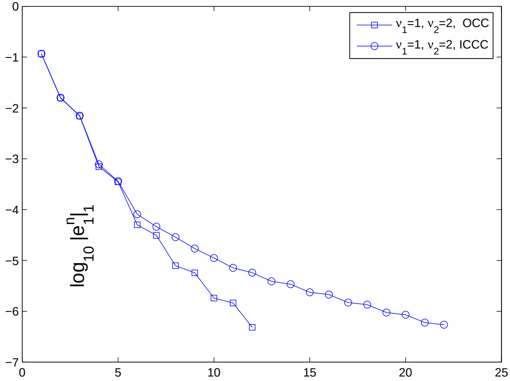

Figure: $\log_{10} |e_1^n|_1$ with respect to the number of iterations. The **OCC** methdos is represent by the blue unfield square and the **ICCC** methods is represent by the blue unfield circle, with $|e_1^n|_1 < 10^{-6}$, $\nu_1 = 1$ and $\nu_2 = 2$.

Conclusion

Ramarks

- When the interface ended with a corner, it is interesting to work (locally) with the Mellin transform.
- It is interesting to work with the **OCC** which bring a better convergence at the corner.

Open Questions

- Study the connection between the interface conditions with optimized coefficients to the corner (**OCC)**, and the interface conditions with constant coefficients up to the corner (**ICCC)**.
- Extension of the results to the three-dimensional case.
- Error estimations.

Thank you for your attention