Diffraction multiple à haute fréquence en acoustique

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X. Antoine, C. Chniti, K. Ramdani. Numerical approximation of high-frequency multiple scattering problems by circular cylinder. Journal of Computational Physics 227 (2008) pp 1754-1771.

Plan

Introduction

The aim of this presentation is to propose a numerical strategy for computing the solution of two dimensional time harmonic acoustic multiple scattering problems at high-frequency.

Equation de Helmholtz

Une onde accoustique $\mathcal{U}$ est une solution de l'équation des ondes :

$$\Delta\mathcal{U} = \frac{1}{c^2}\frac{\partial^2\mathcal{U}}{\partial t^2}$$

Si $\mathcal{U}$ est périodique en temps, de pulsation ω
$\mathcal{U}(x,t) = \mathrm{Re}\,(u(x)\exp(-i\omega t))$
alors l'équation des ondes se réduit à l'équation de Helmholtz

$$\Delta u + k^2 u = 0$$

où $k^2 = \frac{w^2}{c^2}$ est le nombre d'onde.

Onde plane

Ω_- désigne une région bornée de $\mathbb{R}^n$, $n = 2, 3$ de frontière $\partial\Omega$ de classe C^2. Une onde plane $u^{inc}(\mathbf{r}) = e^{ik\boldsymbol{\beta}\cdot\mathbf{r}}$. L'onde incidente plane u^{inc} se diffracte sur Ω_- et génère un champ diffracté u.

Champ total

Le champ total est $u_t = u^{inc} + u_d$.

Problème de diffraction

Le champ diffracté u_d vérifie le problème de diffraction $(\mathcal{E})$:

$$(\mathcal{E}) \quad \begin{cases} \Delta u_d + k^2 u_d = 0 & (\mathbb{R}^n \setminus \overline{\Omega_-}) \\ \Lambda u_d = -\Lambda u^{inc} & (\partial\Omega_-) \\ u \text{ sortant} & \end{cases}$$

où Λ indique l'opérateur à choisir :

- Condition de Dirichlet : $u_t = 0$, i.e $u_d = -u^{inc}$ sur $\partial\Omega_-$
- Condition de Neumann $\frac{\partial u_t}{\partial n} = 0$ i.e $\frac{\partial u_d}{\partial n} = -\frac{\partial u^{inc}}{\partial n}$ sur $\partial\Omega_-$
- Condition de Robin $\frac{\partial u_t}{\partial n} + \alpha u_t = 0$, sur $\partial\Omega_-$, α un nombre complexe ou réel donné(Il est possible de choisir α une fonction).

Theorem

Le problème de diffraction $(\mathcal{E})$ admet une unique solution u.

Amplitude de diffraction : cas où $\Omega_- \subset \mathbb{R}^2$

Proposition

$x = \overrightarrow{OM}$ Il existe une fonction f analytique appelée amplitude de diffraction, telle que :

$$u_d \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} \frac{e^{ik||x||}}{\sqrt{k||x||}} f\left(\frac{x}{||x||}\right), \ quand\ ||x|| \to \infty$$

Diffraction Multiple par M obstacles sphériques

Problème

$$(\mathcal{E}) \quad \begin{cases} \Delta u_d + k^2 u_d = 0 & (\mathbb{R}^n \setminus \overline{\Omega_-}) \\ \Lambda u_d = -\Lambda u^{inc} & (\partial\Omega_-) \\ \lim\limits_{|x|\to+\infty} |\mathbf{x}|^{1/2} \left(\nabla u \cdot \dfrac{\mathbf{x}}{|\mathbf{x}|} - iku \right) = 0. \end{cases}$$

Une idée de résolution

1. Ramener le problème de diffraction multiple à une famille de problèmes de diffraction simple couplés.
2. Obtention d'une famille d'équations différentielles.
3. Réduire le $(\mathcal{E})$ à une seule famille de problèmes de diffraction simple.
4. Obtention d'une solution analytique.

M. Balabane(2004)

Theorem

Let u be the solution of the multiple scattering problem $(\mathcal{E})$. Then, the family of M coupled single scattering problems for $p = 1, \ldots, M$:

$$(\mathcal{E}^p) \quad \begin{cases} \Delta u^p + k^2 u^p = 0 & (\mathbb{R}^2 \setminus \overline{\mathcal{B}_p}) \\ \Lambda u^p = -\Lambda \left(u^{inc} + \sum_{q=1, q\neq p}^{M} u^q \right) & (\partial \mathcal{B}_p) \\ \lim_{|x| \to +\infty} |\mathbf{x}|^{1/2} \left(\nabla u \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - iku \right) = 0. & \end{cases}$$

admits a unique solution $(u^1, \ldots, u^p)$. Furthermore, the following decomposition holds $u = \sum_{p=1}^{M} u^p$.

Two cylinders

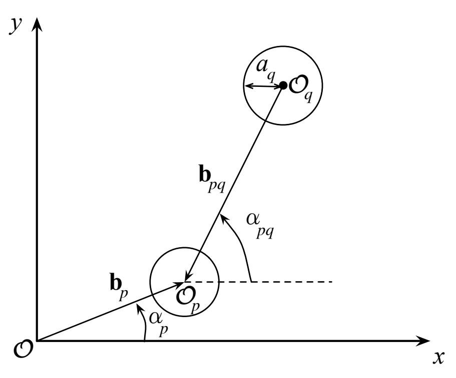


Figure: A view of two typical cylinders.

Let us introduce for all $m \in \mathbb{Z}$ the following cylindrical wavefunctions, which are particular solutions of Helmholtz equations for $r > 0$:

$$\begin{cases} \psi_m(\mathbf{r}) = H_m^{(1)}(kr)e^{im\theta}, \\ \widehat{\psi}_m(\mathbf{r}) = J_m(kr)e^{im\theta}, \end{cases}$$

where J_n is the nth order Bessel function and $H_n^{(1)}$ is the nth order Hankel function of the first kind.
The local cylindrical wavefunctions associated with the scatterer p:

$$\begin{cases} \psi_m^p(\mathbf{r}) = \psi_m(\mathbf{r}_p) = H_m^{(1)}(kr_p)e^{im\theta_p}, \\ \widehat{\psi}_m^p(\mathbf{r}) = \widehat{\psi}_m(\mathbf{r}_p) = J_m(kr_p)e^{im\theta_p}, \end{cases} \qquad \forall\, m \in \mathbb{Z}.$$

u_p is an outgoing solution of a single scattering problem outside a disk.

$$u^p(\mathbf{r}) = \sum_{m\in\mathbb{Z}} c_m^p \psi_m^p(\mathbf{r}), \qquad \forall\, p = 1, \ldots, M,\ \forall r_p > a_p.$$

where the complex coefficients $(c_m^p)_{m\in\mathbb{Z}}$ are determined by imposing the boundary condition on the boundary of the scatterer p:

$$\Lambda u^p = -\Lambda u^{inc} - \sum_{q=1, q\neq p}^{M} \Lambda u^q \qquad \text{on } \partial\mathcal{B}_p.$$

local system of coordinates

$$u^{inc}(\mathbf{r}) = \sum_{m\in\mathbb{Z}} d_m^p \widehat{\psi}_m^p(\mathbf{r}) \text{ where } d_m^p = e^{ik\boldsymbol{\beta}\cdot\mathbf{b}_p}\, e^{im(\frac{\pi}{2}-\beta)}.$$

Theorem (Separation Theorem P.Martin)

Let $1 \leq p, q \leq M$, with $p \neq q$. Then, we have the following relations:

$$\psi_m^q(\mathbf{r}) = \begin{cases} \sum_{n\in\mathbb{Z}} S_{mn}(\mathbf{b}_{pq})\widehat{\psi}_n^p(\mathbf{r}) & \text{for } r_p < b_{pq}, \\ \sum_{n\in\mathbb{Z}} \widehat{S}_{mn}(\mathbf{b}_{pq})\psi_n^p(\mathbf{r}) & \text{for } r_p > b_{pq}, \end{cases} \quad \forall m \in \mathbb{Z},$$

where we have set

$$S_{mn}(\mathbf{b}_{pq}) = \psi_{m-n}(\mathbf{b}_{pq}), \qquad \widehat{S}_{mn}(\mathbf{b}_{pq}) = \widehat{\psi}_{m-n}(\mathbf{b}_{pq}).$$

Unknown Fourier coefficients

Sound-soft case

$\forall m \in \mathbb{Z}, \forall p = 1, \ldots, M$

$$c_m^p + \frac{J_m(ka_p)}{H_m^{(1)}(ka_p)} \sum_{q=1, q\neq p}^{M} \sum_{n \in \mathbb{Z}} S_{nm}(\mathbf{b}_{pq}) c_n^q = -\frac{J_m(ka_p)}{H_m^{(1)}(ka_p)} d_m^p .$$

Sound-hard case

$\forall m \in \mathbb{Z}, \forall p = 1, \ldots, M$

$$c_m^p + \frac{J'_m(ka_p)}{H_m^{(1)'}(ka_p)} \sum_{q=1, q\neq p}^{M} \sum_{n \in \mathbb{Z}} S_{nm}(\mathbf{b}_{pq}) c_n^q = -\frac{J'_m(ka_p)}{H_m^{(1)'}(ka_p)} d_m^p .$$

More compact vector

$$\mathbf{C}^p + \mathbb{D}^p \sum_{q=1, q\neq p}^{M} \left(\mathbb{S}^{p,q}\right)^{\mathsf{T}} \mathbf{C}^q = \mathbf{B}^p \qquad \forall\, p = 1, \ldots, M,$$

where

- $\mathbf{C}^p = (c_n^p)_{n\in\mathbb{Z}}$ is the infinite vector containing the coefficients of the cylindrical decomposition of u^p,
- $(\mathbb{S}^{p,q})^{\mathsf{T}}$ denotes the transpose of the separation matrix $\mathbb{S}^{p,q}$ between the obstacles $\mathcal{B}_p$ and $\mathcal{B}_q$ defined by

$$\mathbb{S}^{p,q} = (\mathbb{S}^{p,q}_{mn})_{m\in\mathbb{Z}, n\in\mathbb{Z}} \qquad \mathbb{S}^{p,q}_{mn} = \psi_{m-n}(\mathbf{b}_{pq}),$$

- $\mathbb{D}^p = (\mathbb{D}^p_{mn})_{mn\in\mathbb{Z}}$ is the diagonal infinite matrix.

$$\mathbb{A}\mathbf{C} = \mathbf{B}$$

where

$$\mathbb{A} = \begin{bmatrix} \mathbb{I} & \mathbb{D}^1\left(\mathbb{S}^{1,2}\right)^{\mathbf{T}} & \dots & \mathbb{D}^1\left(\mathbb{S}^{1,M}\right)^{\mathbf{T}} \\ \mathbb{D}^2\left(\mathbb{S}^{2,1}\right)^{\mathbf{T}} & \mathbb{I} & \dots & \mathbb{D}^2\left(\mathbb{S}^{2,M}\right)^{\mathbf{T}} \\ \vdots & & \ddots & \\ \mathbb{D}^M\left(\mathbb{S}^{M,1}\right)^{\mathbf{T}} & \mathbb{D}^M\left(\mathbb{S}^{M,2}\right)^{\mathbf{T}} & \dots & \mathbb{I} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^1 \\ \mathbf{C}^2 \\ \vdots \\ \mathbf{C}^M \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}^1 \\ \mathbf{B}^2 \\ \vdots \\ \mathbf{B}^M \end{bmatrix}$$

and where $\mathbb{I}$ denotes the identity operator on $\ell^2(\mathbb{C})$.

in the neighborhood of $\mathcal{B}_p$ (namely, for $r_p < \min\limits_{\substack{1\leq q\leq M\\ q\neq p}} b_{pq}$) by the relation

$$u(\mathbf{r}) = \sum_{m\in\mathbb{Z}} c_m^p \psi_m^p(\mathbf{r}) + \sum_{m\in\mathbb{Z}} \left(\sum_{q=1,q\neq p}^{M} \sum_{n\in\mathbb{Z}} S_{nm}(\mathbf{b}_{pq}) c_n^q \right) \widehat{\psi}_m^p(\mathbf{r}).$$

Using $r_p = r - b_p \cos(\theta - \alpha_p) + \mathcal{O}(1/r)$, the scattering amplitude $a(\theta)$ defined by

$$u(\mathbf{r}) = \frac{e^{ikr}}{\sqrt{r}} a(\theta) + \mathcal{O}\left(\frac{1}{r}\right), \qquad \text{as } r \to +\infty.$$

is obtained by

$$a(\theta) = e^{-i\pi/4} \sqrt{\frac{2}{\pi k}} \sum_{p=1}^{p=M} e^{-ib_p k \cos(\theta-\alpha_p)} \left(\sum_{n\in\mathbb{Z}} e^{in(\theta-\frac{\pi}{2})} c_n^p \right).$$

$$\mathcal{C}^p + \mathcal{D}^p \sum_{q=1, q\neq p}^{M} \left(\mathcal{S}^{p,q}\right)^{\mathbf{T}} \mathcal{C}^q = \mathcal{B}^p \qquad \forall\, p = 1, \ldots, M,$$

where

- $\mathcal{C}^p = (c_n^p)_{n=-N_p,\ldots,N_p}$
- $\mathcal{S}^{p,q}$ is the $(2N_p+1)\times(2N_q+1)$ finite dimensional separation matrix $\mathcal{S}^{p,q} = (\mathcal{S}_{mn}^{p,q})_{-N_p\leq m\leq N_p, -N_q\leq n\leq N_q}$ $\qquad \mathcal{S}_{mn}^{p,q} = \psi_{m-n}(\mathbf{b}_{pq}),$
- $\mathcal{D}^p = (\mathcal{D}_{mn}^p)_{-N_p\leq m\leq N_p, -N_q\leq n\leq N_q}$

$$\mathcal{D}_{m,m}^p = \begin{cases} \dfrac{J_m(ka_p)}{H_m^{(1)}(ka_p)} & \text{for sound-soft obstacles,} \\ \dfrac{J_m'(ka_p)}{H_m^{(1)'}(ka_p)} & \text{for sound-hard obstacles,} \end{cases}$$

- $\mathcal{B}^p = -\mathcal{D}^p d^p$, where $d^p = (d_m^p)_{-N_p\leq m\leq N_p}$ is the finite vector containing the $2N_p+1$ first coefficients.

Matrix-sytem

$$\mathcal{A}\mathcal{C} = \mathcal{B}, \quad \mathcal{A} \in \mathbb{C}^{N,N}, \quad N = \sum_{p=1}^{M} (2N_p + 1)$$

$$\mathcal{A} = \begin{bmatrix} \mathcal{I}^1 & \mathcal{D}^1 \left(\mathcal{S}^{1,2}\right)^{\mathbf{T}} & \dots & \mathcal{D}^1 \left(\mathcal{S}^{1,M}\right)^{\mathbf{T}} \\ \mathcal{D}^2 \left(\mathcal{S}^{2,1}\right)^{\mathbf{T}} & \mathcal{I}^2 & \dots & \mathcal{D}^2 \left(\mathcal{S}^{2,M}\right)^{\mathbf{T}} \\ \vdots & & \ddots & \\ \mathcal{D}^M \left(\mathcal{S}^{M,1}\right)^{\mathbf{T}} & \mathcal{D}^M \left(\mathcal{S}^{M,2}\right)^{\mathbf{T}} & \dots & \mathcal{I}^M \end{bmatrix}$$

where $\mathcal{I}^p$ denotes the identity matrix of $\mathbb{C}^{2N_p+1}$ and

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}^1 \\ \mathcal{C}^2 \\ \vdots \\ \mathcal{C}^M \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} \mathcal{B}^1 \\ \mathcal{B}^2 \\ \vdots \\ \mathcal{B}^M \end{bmatrix}$$

Number of modes N_p

1. N_p must be large enough to capture both the propagating and grazing parts of the solution.
2. Taking too many modes for approximating the solution makes the matrix $\mathcal{A}$ illconditioned.

$$N_p = \left[ka_p + \left(\frac{1}{2\sqrt{2}} \ln(2\sqrt{2}\pi ka_p \varepsilon^{-1}) \right)^{\frac{2}{3}} (ka_p)^{1/3} + 1 \right]$$

$\Rightarrow$

$$|c_m^p| \leq \frac{\sqrt{2\sqrt{2}\pi ka_p}}{2} \exp\left(-\sqrt{2}\, ka_p\, \zeta^{3/2}\right)$$

Geometrical configurations

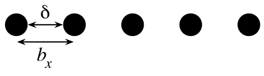


Figure: Regular line with $M = M_x$ disks.

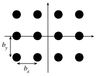


Figure: Regular rectangular lattice with $M = M_x \times M_y$ disks.

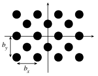


Figure: Triangular lattice with M_x disks on the first row, $(M_x - 1)$ disks on the second one and a total number of M_y odd rows.

Storage

The matrix $\mathcal{A}$ has a particular structure: each one of its off-diagonal blocks is obtained by multiplying the diagonal matrix $\mathcal{D}^p \in \mathbb{C}^{2N_p+1,2N_p+1}$ by the matrix $(\mathcal{S}^{p,q})^{\mathbf{T}} \in \mathbb{C}^{2N_p+1,2N_q+1}$ which has a Toeplitz structure

The storage of $(\mathcal{S}^{p,q})^{\mathbf{T}}$ can be optimized using a compressed version based on the root vector

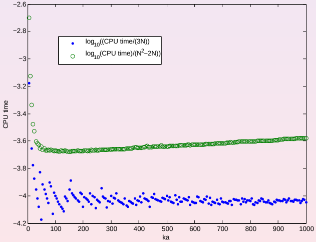
$$\sigma^{p,q} = (\mathcal{S}^{p,q}_{N_q,-Np}, \ldots, \mathcal{S}^{p,q}_{-N_q+1,-N_p}, \mathcal{S}^{p,q}_{-N_q,-N_p}, \ldots, \mathcal{S}^{p,q}_{-N_q,N_p})^{\mathbf{T}}.$$

Comparaion-Storage

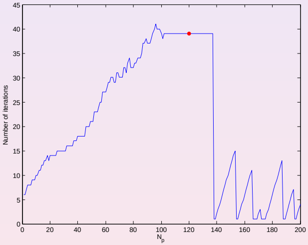
Full version: the $(2N_p+1)(2N_q+2)$ complex coefficients required, the vector root version of $\mathcal{A}$ leads to a memory storage and a CPU time of the order of $\mathcal{O}(4k^2a^2M^2)$

Compressed version: the compressed storage needs $2(2N_p+N_q+1)$ entries, the vector root version of $\mathcal{A}$ leads to a memory storage and a CPU time of the order of $\mathcal{O}(6kaM^2)$

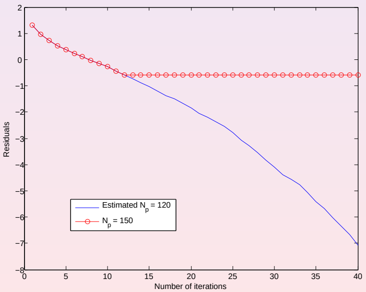
Behaviour of the CPU time according to the wavenumber ka_p for building the matrix $\mathcal{A}$ in the case of the scattering by two circular cylinders fixing N_p by formula



Influence of the order of truncation N_p on the convergence

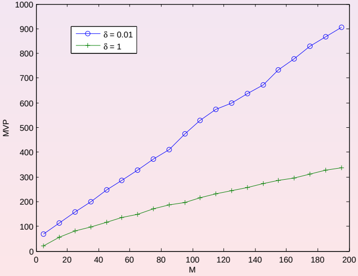


Evolution of the residuals with respect to n^{iter} for $N_p = 120$ and $N_p = 150$.



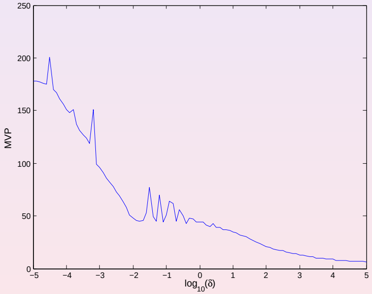
Number of MVPs and obstacles M

Number of MVPs with respect to the number of obstacles M: single-row configuration with $ka = 100$ and two different values of b_x. The GMRES(50) solver is used fixing $tol = 10^{-8}$ (Dirichlet problem).



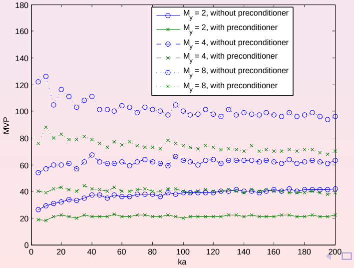
Number of MVPs and distance between two obstacles

Number of MVPs with respect to the distance δ between two obstacles. We fix: $ka = 100$, $M = 10$ obstacles. The solution is obtained with GMRES(50) for $tol = 10^{-8}$ (Dirichlet problem.)



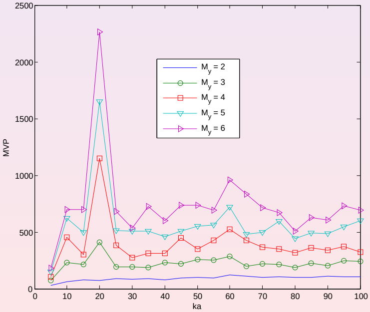
Solution is obtained without and with the preconditioned

Number of MVPs with respect to the wavenumber ka for the rectangular lattice configuration with $bx = 8$ and $by = 13$. We fix $Mx = 8$ and $My = 2, 4, 8$. The solution is obtained without and with the preconditioned GMRES(50) for $tol = 10^{-8}$ (Dirichlet problem).

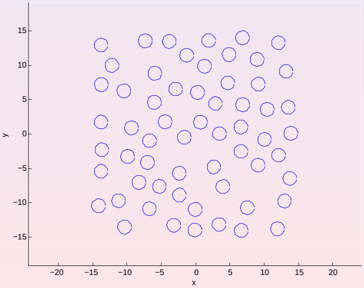


Number of MVPs with respect to the wavenumber *ka*

Number of MVPs with respect to the wavenumber *ka*: rectangular lattice configurations with $Mx = 5, My = 2, \ldots, 6$. We consider $bx = by = 3$ and solve the linear system using the GMRES(50) solver for $tol = 10^{-8}$ (Dirichlet problem).

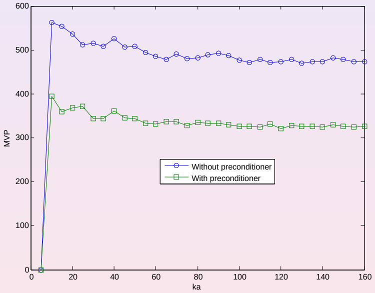


Unstructured configuration



MVPs, wavenumber *ka*, unstructured configuration

Number of MVPs with respect to the wavenumber *ka* for a unstructured configuration (with and without preconditioner).



Conclusion

1. The main difficulty that arises is due to the fact that the complex dense linear system to be solved is very large and ill-conditioned. This is in particular true when the number of scatterers is large and/or for high frequencies.
2. Taking advantage of the particular block Toeplitz structure of the matrix of the linear system, we proposed an adapted storage of the system and an iterative algorithm of resolution, based on a fast MVP computation. We realized a thorough numerical study of the convergence rate with respect to different geometrical parameters of the problem
3. Geometrically-based preconditioner, obtained by taking into account close interactions is tested.

Open questions

1. Etude théorique pour la formule de tranquature.
2. Etude avec les équations intégrales.
3. Etude avec des obstacles non sphériques.
4. Etude dans $\mathbb{R}^3$.
5. Existence d'une plage de fréquences telle que : (Nombre vp = Nombre obstacles). Problème de valeurs propres.
6. Si k est trop petit, trop de valeurs propres : impossibilité de connaître le nombre d'obstacles.
7. Si k est trop grand, perte d'information.
8. Amplitude des plus grandes valeurs propres.