Lecture on semi-classical trace formula

Mouez Dimassi

Université de Bordeaux 1
Départment de mathématiques

Novembre 2011

Outline

Trace formula provide one of the most elegant descriptions of the classical-quantum correspondence. The trace formula is a general identity

$$\sum \text{geometric terms of C, H} = \sum \text{spectral terms of Q H}$$

The spectral terms contain arithmetic information of a fundamental nature (eigenvalues of quantum Hamiltonian). However, they are highly inaccessible. The geometric terms are described in terms of closed orbits of the corresponding classical Hamiltonian.

Trace formula provide one of the most elegant descriptions of the classical-quantum correspondence.The trace formula is a general identity

$$\sum \text{geometric terms of C, H} = \sum \text{spectral terms of Q H}$$

The spectral terms contain arithmetic information of a fundamental nature (eigenvalues of quantum Hamiltonian).However, they are highly inaccessible.The geometric terms are described in terms of closed orbits of the corresponding classical Hamiltonian.

Trace formula provide one of the most elegant descriptions of the classical-quantum correspondence.The trace formula is a general identity

$$\sum \text{geometric terms of C, H} = \sum \text{spectral terms of Q H}$$

The spectral terms contain arithmetic information of a fundamental nature (eigenvalues of quantum Hamiltonian).However, they are highly inaccessible.The geometric terms are described in terms of closed orbits of the corresponding classical Hamiltonian.

Trace formula provide one of the most elegant descriptions of the classical-quantum correspondence.The trace formula is a general identity

$$\sum \text{geometric terms of C, H} = \sum \text{spectral terms of Q H}$$

The spectral terms contain arithmetic information of a fundamental nature (eigenvalues of quantum Hamiltonian).However, they are highly inaccessible.The geometric terms are described in terms of closed orbits of the corresponding classical Hamiltonian.

Trace formula provide one of the most elegant descriptions of the classical-quantum correspondence.The trace formula is a general identity

$$\sum \text{geometric terms of C, H} = \sum \text{spectral terms of Q H}$$

The spectral terms contain arithmetic information of a fundamental nature (eigenvalues of quantum Hamiltonian).However, they are highly inaccessible.The geometric terms are described in terms of closed orbits of the corresponding classical Hamiltonian.

Let $\Gamma = \oplus_{i=1}^{i=n} \mathbb{Z} e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.

Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma \in \Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^* \in \Gamma^*} \widehat{f}(\gamma^*), \ \ f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$: the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n}\mathbb{Z}e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle\gamma, \gamma^*\rangle \in 2\pi\mathbb{Z},\ \forall\gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma\in\Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^*\in\Gamma^*} \widehat{f}(\gamma^*),\ \ f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$: the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n} \mathbb{Z} e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma \in \Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^* \in \Gamma^*} \widehat{f}(\gamma^*), \ \ f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$: the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n}\mathbb{Z}e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle\gamma, \gamma^*\rangle \in 2\pi\mathbb{Z}, \ \forall\gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma\in\Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^*\in\Gamma^*} \widehat{f}(\gamma^*), \quad f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$: the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n}\mathbb{Z}e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma\in\Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^*\in\Gamma^*} \widehat{f}(\gamma^*), \quad f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$: the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n} \mathbb{Z} e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z},\ \forall \gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma\in\Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^*\in\Gamma^*} \widehat{f}(\gamma^*),\ \ f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$. the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

Let $\Gamma = \oplus_{i=1}^{i=n} \mathbb{Z} e_i$ be a lattice in $\mathbb{R}^n$, where $(e_1, ..., e_n)$ is a basis in $\mathbb{R}^n$.
Let $\mathbb{R}^n/\Gamma$ be the flat torus, and

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n; \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma\}.$$

For $f \in S(\mathbb{R}^n)$, the Poisson summation formula may be stated as

$$\sum_{\gamma \in \Gamma} f(\gamma) = \mathrm{vol}(\mathbb{R}^n/\Gamma) \sum_{\gamma^* \in \Gamma^*} \widehat{f}(\gamma^*), \ \ f \in S(\mathbb{R}^n).$$

In the case where $f = F(|x|)$, $\widehat{f} = G(|\xi|)$, the Poisson summation formula is an exact formula relating the

1. $|\gamma|$: the period of the periodic geodesics
2. $|\gamma^*|^2$. the eigenvalues of the $-\Delta$ on the torus $\mathbb{R}^n/\Gamma$.

The Selbergs trace formula is an exact summation formula concerning the case of locally symmetric spaces; this formula was interpreted by H. Huber [Hub59] as a formula relating eigenvalues of the Laplace operator and lengths of closed geodesics (also called the lengths spectrum) on a closed surface of curvature -1.

Consider the classical hamiltonian :

$$H(x,\xi) = \frac{\xi^2 + x^2}{2}, \ (x,\xi) \in \mathbb{R}^2$$

.

Let us denote by

- $$\Phi_t(x,\xi) = \Big(x\cos(t)+\xi\sin(t), -x\sin(t)+\xi\cos(t)\Big) = (x(t),\xi(t))$$

  the classical flow induced by Hamiltons equations

  $$x'(t) = \xi(t), \ \xi'(t) = -x(t), \ \ x(0) = x, \ \xi(0) = \xi.$$

- Fixe $E > 0$, and let

  $$\Sigma_E = \{(x,\xi) \in \mathbb{R}^2; H(x,\xi) = E\}.$$

  The period of the closed orbits $\gamma_{2n\pi}(E)$ on Σ_E are $2n\pi$, $n \in \mathbb{Z}$. (Including repetitions of primitive orbits and

Consider the classical hamiltonian :

$$H(x,\xi) = \frac{\xi^2 + x^2}{2},\ (x,\xi) \in \mathbb{R}^2$$

.

Let us denote by

- $$\Phi_t(x,\xi) = \Big(x\cos(t)+\xi\sin(t), -x\sin(t)+\xi\cos(t)\Big) = (x(t),\xi(t))$$

  the classical flow induced by Hamiltons equations

  $$x'(t) = \xi(t),\ \xi'(t) = -x(t),\ \ x(0) = x,\ \xi(0) = \xi.$$

- Fixe $E > 0$, and let

  $$\Sigma_E = \{(x,\xi) \in \mathbb{R}^2; H(x,\xi) = E\}.$$

  The period of the closed orbits $\gamma_{2n\pi}(E)$ on Σ_E are $2n\pi$, $n \in \mathbb{Z}$. (Including repetitions of primitive orbits and

Consider the classical hamiltonian :

$$H(x,\xi) = \frac{\xi^2 + x^2}{2}, \ (x,\xi) \in \mathbb{R}^2$$

.

Let us denote by

- $$\Phi_t(x,\xi) = \Big(x\cos(t)+\xi\sin(t), -x\sin(t)+\xi\cos(t)\Big) = (x(t),\xi(t))$$

  the classical flow induced by Hamiltons equations

  $$x'(t) = \xi(t), \ \xi'(t) = -x(t), \ \ x(0) = x, \ \xi(0) = \xi.$$

- Fixe $E > 0$, and let

  $$\Sigma_E = \{(x,\xi) \in \mathbb{R}^2; H(x,\xi) = E\}.$$

  The period of the closed orbits $\gamma_{2n\pi}(E)$ on Σ_E are $2n\pi$, $n \in \mathbb{Z}$. (Including repetitions of primitive orbits and

Consider the classical hamiltonian :

$$H(x,\xi) = \frac{\xi^2 + x^2}{2}, \ (x,\xi) \in \mathbb{R}^2$$

.

Let us denote by

- $$\Phi_t(x,\xi) = \Big(x\cos(t)+\xi\sin(t), -x\sin(t)+\xi\cos(t)\Big) = (x(t),\xi(t))$$

  the classical flow induced by Hamiltons equations

  $$x'(t) = \xi(t), \ \xi'(t) = -x(t), \ \ x(0) = x, \ \xi(0) = \xi.$$

- Fixe $E > 0$, and let

  $$\Sigma_E = \{(x,\xi) \in \mathbb{R}^2; H(x,\xi) = E\}.$$

  The period of the closed orbits $\gamma_{2n\pi}(E)$ on Σ_E are $2n\pi$, $n \in \mathbb{Z}$. (Including repetitions of primitive orbits and

- The classical action along $\gamma_{2n\pi}(E) \subset \Sigma_E$:

$$S_\gamma = \int_{\gamma_{2n\pi}(E)} \xi dx = -2n\pi E.$$

- The Maslov index of $\gamma_{2n\pi}(E)$: $\sigma_{2n\pi}(E) = 2n$.

- The classical action along $\gamma_{2n\pi}(E) \subset \Sigma_E$:

$$S_\gamma = \int_{\gamma_{2n\pi}(E)} \xi dx = -2n\pi E.$$

- The Maslov index of $\gamma_{2n\pi}(E)$: $\sigma_{2n\pi}(E) = 2n$.

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2+\xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n+\frac{1}{2}), n = 0,1,2,....$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x-E) = \tilde{\phi}(-x)$. Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\mathrm{tr}\phi(H-E) = \sum_{n=0}^{\infty}\phi(n+\frac{1}{2}-E) = \sum_{n=0}^{\infty}\tilde{\phi}(n+\frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\tilde{\phi}(n+\frac{1}{2}) = \pi\widehat{\phi}(0) + \pi\sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE}e^{in\pi}\widehat{\phi}(2\pi n).$$

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n + \frac{1}{2}), n = 0, 1, 2, ...$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x - E) = \tilde{\phi}(-x)$. Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\mathrm{tr}\phi(H - E) = \sum_{n=0}^{\infty} \phi(n + \frac{1}{2} - E) = \sum_{n=0}^{\infty} \tilde{\phi}(n + \frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\phi}(n + \frac{1}{2}) = \pi\widehat{\tilde{\phi}}(0) + \pi \sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE} e^{in\pi}\widehat{\tilde{\phi}}(2\pi n).$$

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2+\xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n+\frac{1}{2}), n = 0, 1, 2, ...$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x-E) = \tilde{\phi}(-x)$.
Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\mathrm{tr}\phi(H-E) = \sum_{n=0}^{\infty}\phi(n+\frac{1}{2}-E) = \sum_{n=0}^{\infty}\tilde{\phi}(n+\frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\tilde{\phi}(n+\frac{1}{2}) = \pi\widehat{\phi}(0) + \pi\sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE}e^{in\pi}\widehat{\phi}(2\pi n).$$

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n + \frac{1}{2}), n = 0, 1, 2, \ldots.$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x - E) = \tilde{\phi}(-x)$. Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\mathrm{tr}\phi(H - E) = \sum_{n=0}^{\infty} \phi(n + \frac{1}{2} - E) = \sum_{n=0}^{\infty} \tilde{\phi}(n + \frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\phi}(n + \frac{1}{2}) = \pi\widehat{\tilde{\phi}}(0) + \pi \sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE} e^{in\pi}\widehat{\tilde{\phi}}(2\pi n).$$

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n + \frac{1}{2}), n = 0, 1, 2, \dots.$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x - E) = \tilde{\phi}(-x)$. Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\operatorname{tr}\phi(H - E) = \sum_{n=0}^{\infty} \phi(n + \frac{1}{2} - E) = \sum_{n=0}^{\infty} \tilde{\phi}(n + \frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\phi}(n + \frac{1}{2}) = \pi\widehat{\tilde{\phi}}(0) + \pi \sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE} e^{in\pi} \widehat{\tilde{\phi}}(2\pi n).$$

Consider the one-dimensional harmonic oscillator corresponding to the classical Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2+\xi^2)$:

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The eigenvalues of $\widehat{H}$ are :

$$E_n(h) = (n+\frac{1}{2}), n = 0, 1, 2, \ldots.$$

Let $\phi \in S(\mathbb{R})$. Assume that $\tilde{\phi}(x) := \phi(x-E) = \tilde{\phi}(-x)$. Applying the Poisson formula to $\tilde{\phi}(x)$, we obtain:

$$\mathrm{tr}\phi(H-E) = \sum_{n=0}^{\infty} \phi(n+\frac{1}{2}-E) = \sum_{n=0}^{\infty} \tilde{\phi}(n+\frac{1}{2})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\phi}(n+\frac{1}{2}) = \pi\widehat{\tilde{\phi}}(0) + \pi \sum_{n\in\mathbb{Z}^*} e^{-i2\pi nE} e^{in\pi}\widehat{\tilde{\phi}}(2\pi n).$$

In the above formulas, such as the original Selberg trace formula, the identities are exact, while in general they hold only in semi-classical or high-energy : called "‘semi-classical Gutzwiler trace formula"’ limits.

In the above formulas, such as the original Selberg trace formula, the identities are exact, while in general they hold only in semi-classical or high-energy : called "‘semi-classical Gutzwiler trace formula"’ limits.

Classical Hamiltonian

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a classical Hamiltonian. We will denote by X_H the Hamiltonian vector field

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j}\partial_{x_j} - \frac{\partial H}{\partial x_j}\partial_{\xi_j}$$

. Let us denote by Φ_t the classical flow induced by Hamiltons equations with Hamiltonian H, i.e., $\Phi_t(x,\xi) = (x(t), \xi(t))$, where

$$x'(t) = \frac{\partial H}{\partial \xi}(x(t), \xi(t)),\ \xi'(t) = -\frac{\partial H}{\partial xi}(x(t), \xi(t)), x(0) = x, \xi(0) = \xi.$$

Notice that, if

$$\Sigma_E := \{(x,\xi) \in \mathbb{R}^{2n}; H(x,\xi) = E\}$$

is a compact set, then the hamilton flow Φ_t restricted to Σ_E is complete. This follows from the preservation of H by the dynamics, i.e. $H(\Phi_t(x,\xi)) = H(x,\xi)$.

Classical Hamiltonian

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a classical Hamiltonian. We will denote by X_H the Hamiltonian vector field

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j} \partial_{x_j} - \frac{\partial H}{\partial x_j} \partial_{\xi_j}$$

. Let us denote by Φ_t the classical flow induced by Hamiltons equations with Hamiltonian H, i.e., $\Phi_t(x, \xi) = (x(t), \xi(t))$, where

$$x'(t) = \frac{\partial H}{\partial \xi}(x(t), \xi(t)),\ \xi'(t) = -\frac{\partial H}{\partial xi}(x(t), \xi(t)), x(0) = x, \xi(0) = \xi.$$

Notice that, if

$$\Sigma_E := \{(x, \xi) \in \mathbb{R}^{2n}; H(x, \xi) = E\}$$

is a compact set, then the hamilton flow Φ_t restricted to Σ_E is complete. This follows from the preservation of H by the dynamics, i.e. $H(\Phi_t(x, \xi)) = H(x, \xi)$.

Classical Hamiltonian

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a classical Hamiltonian. We will denote by X_H the Hamiltonian vector field

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j}\partial_{x_j} - \frac{\partial H}{\partial x_j}\partial_{\xi_j}$$

. Let us denote by Φ_t the classical flow induced by Hamiltons equations with Hamiltonian H, i.e., $\Phi_t(x,\xi) = (x(t),\xi(t))$, where

$$x'(t) = \frac{\partial H}{\partial \xi}(x(t),\xi(t)),\ \xi'(t) = -\frac{\partial H}{\partial xi}(x(t),\xi(t)), x(0) = x, \xi(0) = \xi.$$

Notice that, if

$$\Sigma_E := \{(x,\xi) \in \mathbb{R}^{2n}; H(x,\xi) = E\}$$

is a compact set, then the hamilton flow Φ_t restricted to Σ_E is complete. This follows from the preservation of H by the dynamics, i.e. $H(\Phi_t(x,\xi)) = H(x,\xi)$

Classical Hamiltonian

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a classical Hamiltonian. We will denote by X_H the Hamiltonian vector field

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j} \partial_{x_j} - \frac{\partial H}{\partial x_j} \partial_{\xi_j}$$

. Let us denote by Φ_t the classical flow induced by Hamiltons equations with Hamiltonian H, i.e., $\Phi_t(x, \xi) = (x(t), \xi(t))$, where

$$x'(t) = \frac{\partial H}{\partial \xi}(x(t), \xi(t)),\ \xi'(t) = -\frac{\partial H}{\partial xi}(x(t), \xi(t)), x(0) = x, \xi(0) = \xi.$$

Notice that, if

$$\Sigma_E := \{(x, \xi) \in \mathbb{R}^{2n}; H(x, \xi) = E\}$$

is a compact set, then the hamilton flow Φ_t restricted to Σ_E is complete. This follows from the preservation of H by the dynamics, i.e., $H(\Phi_t(x, \xi)) = H(x, \xi)$.

Classical Hamiltonian

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a classical Hamiltonian. We will denote by X_H the Hamiltonian vector field

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j} \partial_{x_j} - \frac{\partial H}{\partial x_j} \partial_{\xi_j}$$

. Let us denote by Φ_t the classical flow induced by Hamiltons equations with Hamiltonian H, i.e., $\Phi_t(x, \xi) = (x(t), \xi(t))$, where

$$x'(t) = \frac{\partial H}{\partial \xi}(x(t), \xi(t)),\ \xi'(t) = -\frac{\partial H}{\partial xi}(x(t), \xi(t)), x(0) = x, \xi(0) = \xi.$$

Notice that, if

$$\Sigma_E := \{(x, \xi) \in \mathbb{R}^{2n}; H(x, \xi) = E\}$$

is a compact set, then the hamilton flow Φ_t restricted to Σ_E is complete. This follows from the preservation of H by the dynamics, i.e. $H(\Phi_t(x, \xi)) = H(x, \xi)$.

Closed Orbits

Definition

1. A closed orbit (γ, T) of the Hamiltonian H consists of an orbit of X_H which is homeomorphic to a circle and a nonzero real number T so that $\Phi_T(z) = z$ for all $z \in \gamma$.
2. T will be called the period of γ.
3. We will denote by $T^*(\gamma)) > 0$ (the primitive period) the smallest $T > 0$ for which $\Phi_T(z) = z$.

Closed Orbits

Definition

1. A closed orbit (γ, T) of the Hamiltonian H consists of an orbit of X_H which is homeomorphic to a circle and a nonzero real number T so that $\Phi_T(z) = z$ for all $z \in \gamma$.
2. T will be called the period of γ.
3. We will denote by $T^*(\gamma)) > 0$ (the primitive period) the smallest $T > 0$ for which $\Phi_T(z) = z$.

Closed Orbits

Definition

1. A closed orbit (γ, T) of the Hamiltonian H consists of an orbit of X_H which is homeomorphic to a circle and a nonzero real number T so that $\Phi_T(z) = z$ for all $z \in \gamma$.
2. T will be called the period of γ.
3. We will denote by $T^*(\gamma)) > 0$ (the primitive period) the smallest $T > 0$ for which $\Phi_T(z) = z$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

Linear Poincaré map of a closed orbit

Definition

Let (γ, T) with $H(\gamma) = E$ be a closed orbit. We restrict the flow to $\Sigma_E := \{H = E\}$ and take an hypersurface S_γ inside Σ_E transversal to γ at the point z_0. Let $z \in S_\gamma$ near z_0. Then $\Phi_t(z)$ intersect S_γ at time $T_z \sim T^*_{z_0}$. The associated return map P_γ is the map $P_\gamma : S_\gamma \ni z \to \Phi_{T_z}(z)$. It's a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincare map, an inversible (symplectic) endomorphism of the tangent space $T_{z_0}S$.

non degenerate closed orbit

Definition

A closed orbit will called non degenerate if 1 is not an eigenvalue of the Poincaré map Π_γ. .

Maslov Index

Admissible symbols

Definition

We say that $m : \mathbb{R}^n \to [0, +\infty[$ is an order function if there are constants $C_0 > 0$, $N_0 > 0$ such that

$$m(x) \leq C_0 \langle x - y \rangle^{N_0} m(y).$$

Definition

Let m be an order function on $\mathbb{R}^n$. We let $S(\mathbb{R}^n, m) = S(m)$ be the set of $a \in C^\infty(\mathbb{R}^n)$ such that for every $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$, such that $|\partial^\alpha a(x)| \leq C_\alpha m(x)$. In particular $S(1)$ is the space of symbols which are bounded with all their derivatives.

Admissible symbols

Definition

We say that $m : \mathbb{R}^n \to [0, +\infty[$ is an order function if there are constants $C_0 > 0$, $N_0 > 0$ such that

$$m(x) \leq C_0 \langle x - y \rangle^{N_0} m(y).$$

Definition

Let m be an order function on $\mathbb{R}^n$. We let $S(\mathbb{R}^n, m) = S(m)$ be the set of $a \in C^\infty(\mathbb{R}^n)$ such that for every $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$, such that $|\partial^\alpha a(x)| \leq C_\alpha m(x)$. In particular $S(1)$ is the space of symbols which are bounded with all their derivatives.

For $p \in S(m)$, we use the Weyl quantization :

$$p^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\xi/h} p(\frac{(x+y)}{2}, \xi) dx d\xi.$$

$u \in S(\mathbb{R}^n)$.

For $p \in S(m)$, we use the Weyl quantization :

$$p^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\xi/h} p(\frac{(x+y)}{2}, \xi) dx d\xi.$$

$u \in S(\mathbb{R}^n)$.

Theorem

Let $m \geq 1$ be an order function, and let $p \in S(m)$ be real-valued. We assume that $(p+i)$ is elliptic. Let $a < b$ such that $p^{-1}([a,b])$ is compacat. Then for $f \in C_0^\infty(\mathbb{R})$, $f(p^w(x, hD_x))$ is trace class, and

$$\operatorname{tr}\Big(f(p^w(x, hD_x h)\Big) \sim \sum_j b_j(f) h^j,$$

with

$$b_0 = \frac{1}{(2\pi)^n} \int f(p(x,\xi)) dx d\xi.$$

$$b_j(f) = \iint \sum_{l \geq 1}^{N_j} f^{(l)}(p(x,\xi)) P_{j,l}(x,\xi) dx d\xi.$$

Theorem

Let $m \geq 1$ be an order function, and let $p \in S(m)$ be real-valued. We assume that $(p+i)$ is elliptic. Let $a < b$ such that $p^{-1}([a,b])$ is compacat.Then for $f \in C_0^\infty(\mathbb{R})$, $f(p^w(x, hD_x))$ is trace class, and

$$\mathrm{tr}\Big(f(p^w(x, hD_x h)\Big) \sim \sum_j b_j(f) h^j,$$

with

$$b_0 = \frac{1}{(2\pi)^n} \int f(p(x,\xi)) dx d\xi.$$

$$b_j(f) = \iint \sum_{l \geq 1}^{N_j} f^{(l)}(p(x,\xi)) P_{j,l}(x,\xi) dx d\xi.$$

Weyl term

Theorem

Let $H \in S(1)$ be real valued. We assume that $H^{-1}[E-\delta(E), E+\delta(E)]$ is compact, and that $\nabla H \neq 0$ on $H^{-1}[E-\delta(E), E+\delta(E)]$. There exists $C > 0$ large enough such that for $\widehat{g} \in C_0^\infty(]-\frac{1}{C}, \frac{1}{C}[;\mathbb{R})$ and $f \in C_0^\infty(]E-\delta(E), E+\delta(E)[;\mathbb{R})$, we have :

$$D_{f,\theta}^{\mathrm{Weyl}}(\lambda) := \mathrm{tr}\Big(f(H)\theta(\frac{\lambda - H}{h})\Big) \sim (2\pi h)^{1-n} \sum_{j=0}^{\infty} h^j a_j(\lambda), \ (h \searrow 0).$$

$$a_0(\lambda) = \widehat{g}(0) f(\lambda) \int_{\Sigma_\lambda} \frac{d\Sigma_\lambda}{|\nabla H|},$$

$(d\Sigma_\lambda$ is the Euclidean measure on $\Sigma_\lambda)$

Weyl term

Theorem

Let $H \in S(1)$ be real valued. We assume that $H^{-1}[E-\delta(E), E+\delta(E)]$ is compact, and that $\nabla H \neq 0$ on $H^{-1}[E-\delta(E), E+\delta(E)]$. There exists $C > 0$ large enough such that for $\widehat{g} \in C_0^\infty(]-\frac{1}{C}, \frac{1}{C}[;\mathbb{R})$ and $f \in C_0^\infty(]E-\delta(E), E+\delta(E)[;\mathbb{R})$, we have :

$$D_{f,\theta}^{\mathrm{Weyl}}(\lambda) := \mathrm{tr}\Big(f(H)\theta(\frac{\lambda-H}{h})\Big) \sim (2\pi h)^{1-n}\sum_{j=0}^{\infty} h^j a_j(\lambda), \ (h \searrow 0).$$

$$a_0(\lambda) = \widehat{g}(0)f(\lambda)\int_{\Sigma_\lambda}\frac{d\Sigma_\lambda}{|\nabla H|},$$

($d\Sigma_\lambda$ is the Euclidean measure on Σ_λ)

Weyl term

Theorem

Let $H \in S(1)$ be real valued. We assume that $H^{-1}[E-\delta(E), E+\delta(E)]$ is compact, and that $\nabla H \neq 0$ on $H^{-1}[E-\delta(E), E+\delta(E)]$. There exists $C > 0$ large enough such that for $\widehat{g} \in C_0^\infty(]-\frac{1}{C}, \frac{1}{C}[;\mathbb{R})$ and $f \in C_0^\infty(]E-\delta(E), E+\delta(E)[;\mathbb{R})$, we have :

$$D_{f,\theta}^{\mathrm{Weyl}}(\lambda) := \mathrm{tr}\Big(f(H)\theta(\frac{\lambda - H}{h})\Big) \sim (2\pi h)^{1-n} \sum_{j=0}^{\infty} h^j a_j(\lambda), \ (h \searrow 0).$$

$$a_0(\lambda) = \widehat{g}(0) f(\lambda) \int_{\Sigma_\lambda} \frac{d\Sigma_\lambda}{|\nabla H|},$$

($d\Sigma_\lambda$ is the Euclidean measure on Σ_λ)

Weyl term

Theorem

Let $H \in S(1)$ be real valued. We assume that $H^{-1}[E-\delta(E), E+\delta(E)]$ is compact, and that $\nabla H \neq 0$ on $H^{-1}[E-\delta(E), E+\delta(E)]$. There exists $C>0$ large enough such that for $\widehat{g} \in C_0^\infty(]-\frac{1}{C}, \frac{1}{C}[;\mathbb{R})$ and $f \in C_0^\infty(]E-\delta(E), E+\delta(E)[;\mathbb{R})$, we have :

$$D_{f,\theta}^{\mathrm{Weyl}}(\lambda) := \mathrm{tr}\Big(f(H)\theta(\frac{\lambda-H}{h})\Big) \sim (2\pi h)^{1-n}\sum_{j=0}^{\infty} h^j a_j(\lambda), \;\; (h \searrow 0).$$

$$a_0(\lambda) = \widehat{g}(0)f(\lambda)\int_{\Sigma_\lambda}\frac{d\Sigma_\lambda}{|\nabla H|},$$

($d\Sigma_\lambda$ is the Euclidean measure on Σ_λ)

Weyl term

Theorem

Let $H \in S(1)$ be real valued. We assume that $H^{-1}[E-\delta(E), E+\delta(E)]$ is compact, and that $\nabla H \neq 0$ on $H^{-1}[E-\delta(E), E+\delta(E)]$. There exists $C > 0$ large enough such that for $\widehat{g} \in C_0^\infty(]-\frac{1}{C}, \frac{1}{C}[;\mathbb{R})$ and $f \in C_0^\infty(]E-\delta(E), E+\delta(E)[;\mathbb{R})$, we have :

$$D_{f,\theta}^{\mathrm{Weyl}}(\lambda) := \mathrm{tr}\Big(f(H)\theta(\frac{\lambda - H}{h})\Big) \sim (2\pi h)^{1-n} \sum_{j=0}^{\infty} h^j a_j(\lambda), \;(h \searrow 0).$$

$$a_0(\lambda) = \widehat{g}(0) f(\lambda) \int_{\Sigma_\lambda} \frac{d\Sigma_\lambda}{|\nabla H|},$$

($d\Sigma_\lambda$ is the Euclidean measure on Σ_λ)

Weyl term

Theorem

let $p \in S(1)$ be real-valued. Let $a < b$ such that $\nabla p \neq 0$ on $p^{-1}(\{a, b\})$. We assume that $p^{-1}(a - o(1), b + o(1)]$ is compact. Then

$$N_{[a,b]}(h) = (2\pi h)^{-n}\mathrm{vol}(p^{-1}([a, b])) + O(h^{-n+1}).$$

Assumptions

Let $\widehat{H} := H^w(x, hD_x)$, where $H \in S(1)$ is real-valued. Let $(\Gamma_E)_T$ be the set of all periodic orbits on $\sum_E$ with periods T_γ, $0 < |T_\gamma| \leq T$ (including repetitions of primitive orbits and assigning negative periods to primitive orbits traced in the opposite sense).

Assumptions

1. (H.1) There exists $\delta E > 0$ such that $H^{-1}([E - \delta E, E + \delta E])$ is a compact set of $\mathbb{R}^{2n}$ and E is a noncritical value of H (i.e. $H(z) = E \to \nabla H(z) \neq 0$).
2. (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.
3. (H.3) All γ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding "Poincaré map", P_γ.
We can now state the Gutzwiler trace formula. Let $\widehat{A} = Op_\hbar^w(A)$ be a quantum observable, such that A satisfies the following
4. (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_z^\gamma A(z)| \leq C_\gamma < z >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

5. (H.5) g a C^∞ function whose Fourier transform $\widehat{g}$ is of

Assumptions

1. (H.1) There exists $\delta E > 0$ such that $H^{-1}([E - \delta E, E + \delta E])$ is a compact set of $\mathbb{R}^{2n}$ and E is a noncritical value of H (i.e. $H(z) = E \to \nabla H(z) \neq 0$).
2. (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.
3. (H.3) All γ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding "Poincaré map", P_γ.
   We can now state the Gutzwiller trace formula. Let $\widehat{A} = Op_\hbar^w(A)$ be a quantum observable, such that A satisfies the following
4. (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_z^\gamma A(z)| \leq C_\gamma < z >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

5. (H.5) g a C^∞ function whose Fourier transform $\widehat{g}$ is of

Assumptions

1. (H.1) There exists $\delta E > 0$ such that $H^{-1}([E-\delta E, E+\delta E])$ is a compact set of $\mathbb{R}^{2n}$ and E is a noncritical value of H (i.e. $H(z) = E \rightarrow \nabla H(z) \neq 0$).
2. (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.
3. (H.3) All γ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding "Poincaré map", P_γ. We can now state the Gutzwiller trace formula. Let $\widehat{A} = Op_\hbar^w(A)$ be a quantum observable, such that A satisfies the following
4. (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_z^\gamma A(z)| \leq C_\gamma < z >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

5. (H.5) g a $\mathcal{C}^\infty$ function whose Fourier transform $\widehat{g}$ is of

Assumptions

1. (H.1) There exists $\delta E > 0$ such that $H^{-1}([E - \delta E, E + \delta E])$ is a compact set of $\mathbb{R}^{2n}$ and E is a noncritical value of H (i.e. $H(z) = E \to \nabla H(z) \neq 0$).
2. (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.
3. (H.3) All γ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding "Poincaré map", P_γ.

   We can now state the Gutzwiler trace formula. Let $\widehat{A} = Op_\hbar^w(A)$ be a quantum observable, such that A satisfies the following
4. (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_z^\gamma A(z)| \leq C_\gamma < z >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

5. (H.5) g a C^∞ function whose Fourier transform $\widehat{g}$ is of

Assumptions

1. (H.1) There exists $\delta E > 0$ such that $H^{-1}([E - \delta E, E + \delta E])$ is a compact set of $\mathbb{R}^{2n}$ and E is a noncritical value of H (i.e. $H(z) = E \to \nabla H(z) \neq 0$).
2. (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.
3. (H.3) All γ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding "Poincaré map", P_γ.
   We can now state the Gutzwiler trace formula. Let $\widehat{A} = Op_\hbar^w(A)$ be a quantum observable, such that A satisfies the following
4. (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_z^\gamma A(z)| \leq C_\gamma < z >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

5. (H.5) g a $\mathcal{C}^\infty$ function whose Fourier transform $\widehat{g}$ is of

Assumptions

Let χ be a smooth function with a compact support contained in $]E-\delta E, E+\delta E[$, equal to 1 in a neighborhood of E.
Then the following "regularized density of states" $\rho_A(E)$ is well defined:

$$\rho_A(E) = \mathrm{Tr}\left(\chi(\widehat{H})\widehat{A}\chi(\widehat{H})g\left(\frac{E-\widehat{H}}{\hbar}\right)\right)$$

Note that (H.1) implies that the spectrum of $\widehat{H}$ is purely discrete in a neighborhood of E so that $\rho_A(E)$ is well defined.
Then we have the following:

Assumptions

Let χ be a smooth function with a compact support contained in $]E - \delta E, E + \delta E[$, equal to 1 in a neighborhood of E.
Then the following "regularized density of states" $\rho_A(E)$ is well defined:

$$\rho_A(E) = \mathrm{Tr}\left(\chi(\widehat{H})\widehat{A}\chi(\widehat{H})g\left(\frac{E-\widehat{H}}{\hbar}\right)\right)$$

Note that (H.1) implies that the spectrum of $\widehat{H}$ is purely discrete in a neighborhood of E so that $\rho_A(E)$ is well defined.
Then we have the following:

Assumptions

Let χ be a smooth function with a compact support contained in $]E-\delta E, E+\delta E[$, equal to 1 in a neighborhood of E.
Then the following “regularized density of states” $\rho_A(E)$ is well defined:

$$\rho_A(E) = \mathrm{Tr}\left(\chi(\widehat{H})\widehat{A}\chi(\widehat{H})g\left(\frac{E-\widehat{H}}{\hbar}\right)\right)$$

Note that (H.1) implies that the spectrum of $\widehat{H}$ is purely discrete in a neighborhood of E so that $\rho_A(E)$ is well defined.
Then we have the following:

Assumptions

Let χ be a smooth function with a compact support contained in $]E-\delta E, E+\delta E[$, equal to 1 in a neighborhood of E.
Then the following "regularized density of states" $\rho_A(E)$ is well defined:

$$\rho_A(E) = \mathrm{Tr}\left(\chi(\widehat{H})\widehat{A}\chi(\widehat{H})g\left(\frac{E-\widehat{H}}{\hbar}\right)\right)$$

Note that (H.1) implies that the spectrum of $\widehat{H}$ is purely discrete in a neighborhood of E so that $\rho_A(E)$ is well defined.
Then we have the following:

Theorem

Assume (H.0)-(H.3) *are satified for H,* (H.4) *for A and* (H.5) *for g. Then the following asymptotic expansion holds true, modulo* $O(h^\infty)$,

$$\rho_A(E) \equiv D^{\mathrm{Weyl}}_{g,\chi}(E) + \sum_{\gamma\in(\Gamma_E)_T} D_\gamma(E).$$

$$D^{\mathrm{Weyl}}_{g,\chi}(E) = \widehat{g}(0)(2\pi h)^{-(n-1)} \int_{\Sigma_E} A(\alpha)d\sigma_E(\alpha) + \sum_{k\geq -n+2} c_k(\widehat{g})h^k$$

$$D_\gamma(E) = (2\pi)^{n/2-1}\{\widehat{g}(T_\gamma)\frac{e^{i(S_\gamma/h+\sigma_\gamma\pi/2)}}{|\det(I-P_\gamma)|^{1/2}}\int_0^{T^*_\gamma} A(\alpha_s)ds$$

$$+\sum_{j\geq 1} d^\gamma_j(\widehat{g})h^j$$

Theorem

Assume (H.0)-(H.3) *are satified for* H*,* (H.4) *for* A *and* (H.5) *for* g*. Then the following asymptotic expansion holds true, modulo* $O(h^\infty)$*,*

$$\rho_A(E) \equiv D_{g,\chi}^{\mathrm{Weyl}}(E) + \sum_{\gamma\in(\Gamma_E)_T} D_\gamma(E).$$

$$D_{g,\chi}^{\mathrm{Weyl}}(E) = \widehat{g}(0)(2\pi h)^{-(n-1)} \int_{\Sigma_E} A(\alpha)d\sigma_E(\alpha) + \sum_{k\geq -n+2} c_k(\widehat{g})h^k$$

$$D_\gamma(E) = (2\pi)^{n/2-1}\{\widehat{g}(T_\gamma)\frac{e^{i(S_\gamma/h+\sigma_\gamma\pi/2)}}{|\det(I-P_\gamma)|^{1/2}}\int_0^{T_\gamma^*} A(\alpha_s)ds$$

$$+\sum_{j\geq 1} d_j^\gamma(\widehat{g})h^j$$

Theorem

Assume (H.0)-(H.3) *are satified for* H, (H.4) *for* A *and* (H.5) *for* g. *Then the following asymptotic expansion holds true, modulo* $O(h^\infty)$,

$$\rho_A(E) \equiv D^{\mathrm{Weyl}}_{g,\chi}(E) + \sum_{\gamma\in(\Gamma_E)_T} D_\gamma(E).$$

$$D^{\mathrm{Weyl}}_{g,\chi}(E) = \widehat{g}(0)(2\pi h)^{-(n-1)} \int_{\Sigma_E} A(\alpha)d\sigma_E(\alpha) + \sum_{k\geq -n+2} c_k(\widehat{g})h^k$$

$$D_\gamma(E) = (2\pi)^{n/2-1}\{\widehat{g}(T_\gamma)\frac{e^{i(S_\gamma/h+\sigma_\gamma\pi/2)}}{|\det(I-P_\gamma)|^{1/2}}\int_0^{T^*_\gamma} A(\alpha_s)ds$$

$$+\sum_{j\geq 1} d_j^\gamma(\widehat{g})h^j$$

Theorem

Assume (H.0)-(H.3) *are satified for H,* (H.4) *for A and* (H.5) *for g. Then the following asymptotic expansion holds true, modulo* $O(h^\infty)$,

$$\rho_A(E) \equiv D^{\mathrm{Weyl}}_{g,\chi}(E) + \sum_{\gamma\in(\Gamma_E)_T} D_\gamma(E).$$

$$D^{\mathrm{Weyl}}_{g,\chi}(E) = \widehat{g}(0)(2\pi h)^{-(n-1)} \int_{\Sigma_E} A(\alpha)d\sigma_E(\alpha) + \sum_{k\geq -n+2} c_k(\widehat{g})h^k$$

$$D_\gamma(E) = (2\pi)^{n/2-1}\{\widehat{g}(T_\gamma)\frac{e^{i(S_\gamma/h+\sigma_\gamma\pi/2)}}{|\det(I-P_\gamma)|^{1/2}}\int_0^{T^*_\gamma} A(\alpha_s)ds$$

$$+\sum_{j\geq 1} d_j^\gamma(\widehat{g})h^j$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T_γ^* is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T_γ^* is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T^*_γ is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T_γ^* is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T_γ^* is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T_γ^* is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

semi-classical Gutzwiler trace formula

where

1. $A(\alpha)$ is the classical Weyl symbol of $\widehat{A}$,
2. T^*_γ is the primitive period of γ,
3. σ_γ is the Maslov index of γ ($\sigma_\gamma \in \mathbb{Z}$),
4. $S_\gamma = \oint_\gamma pdq$ is the classical action along γ,
5. $c_k(\widehat{g})$ are distributions in $\widehat{g}$ with support in $\{0\}$,
6. $d_j^\gamma(\widehat{g})$ are distributions in $\widehat{g}$ with support $\{T_\gamma\}$
7. $d\sigma_E$ is the Liouville measure on $\sum_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \qquad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$