

# Energy critical heat equation in two space dimensions

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# Introduction

Consider the IVP

$$\begin{cases} u_t - \Delta u = f(u) \\ u(0) = u_0 \end{cases} \quad (1)$$

- $u(t, x) : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ .
- $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  with  $f(0) = 0$ .
- $u_0 \in L^\infty \implies \exists T(u_0) > 0$  and a unique solution  $u \in \mathcal{C}([0, T[; L^\infty)$  to (1).
- The Cauchy problem (1) has been extensively studied in the scale of Lebesgue spaces  $L^q$ , especially for polynomial type nonlinearities i.e.

$$f(u) := \pm |u|^{\gamma-1} u, \quad \gamma > 1. \quad (2)$$

- Scaling invariance:

$$u(t, x) \implies u_\lambda(t, x) := \lambda^{2/\gamma} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

- The Lebesgue space  $L^{q_c}(\mathbb{R}^d)$  with index  $q_c := \frac{d(\gamma-1)}{2}$  is the only one invariant under the same scaling.
- **Subcritical case:**  $q \geq q_c \geq 1$ 
  - **Weissler** [1980]: Existence and uniqueness in  $\mathcal{C}_T(L^q) \cap L_{loc}^\infty(L^\infty)$ .
  - **Brezis-Cazenave** [1996]: Unconditional uniqueness.
- **Critical case:**  $q = q_c$  and  $d \geq 3$ . There are two sub-cases:
  - $q_c > \gamma + 1$ , or equivalently  $\gamma > \frac{d+2}{d-2}$ . The existence was proved by **Weissler** and the uniqueness by **Brezis-Cazenave**.
  - $q = q_c = \gamma + 1$ , or equivalently  $q = \frac{2d}{d-2}$  and  $\gamma = \frac{d+2}{d-2}$  (**double critical case**).
    - **Weissler** [1981]: Conditional wellposedness.
    - **Ni-Sacks** [1985]: Nonuniqueness where the underlying space is the unit ball.
    - **Terraneo** [2002]: Nonuniqueness for the whole space and for suitable data.
    - **Matos-Terraneo** [2003]: Nonuniqueness for general data.

- **Supercritical case:**  $q < q_c$ .
  - **Haraux-Weissler** [1982]: Uniqueness is lost for initial data  $u_0 = 0$  and for  $1 + \frac{1}{d} < \gamma < \frac{d+2}{d-2}$ .
  - There exists no (local) solution in any reasonable weak sense.
- **Ribaud** [1998]: Wellposedness in Sobolev spaces  $H_p^s$ .
- **Miao-Zhang** [2004]: Wellposedness in Besov spaces.

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# Two dimensional case

- One of the results of [Miao-Zhang](#) is the global well posedness in  $H^1(\mathbb{R}^2)$  for small data, and for nonlinearities  $f$  satisfying

$$|f'(u)| \leq C|u|^2 e^u .$$

- Recall that the Sobolev space  $H^1(\mathbb{R}^2)$  is embedded in all Lebesgue spaces  $L^p(\mathbb{R}^2)$  for every  $2 \leq p < \infty$  but not in  $L^\infty(\mathbb{R}^2)$ .
- The optimal (critical) Sobolev embedding is known to be

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2), \quad (3)$$

where  $\mathcal{L}(\mathbb{R}^2)$  is the Orlicz space associated to the function  $\phi(s) = e^{s^2} - 1$

- A natural question to ask then is: what is the [critical](#) nonlinearity in dimension  $d = 2$ ?

A natural model to investigate in 2D is

$$\begin{cases} u_t - \Delta u = \pm u(e^{u^2} - 1) & \text{in } \mathbb{R}^2 \\ u(0) = u_0 \end{cases} \quad (4)$$

- **Goal:** Existence and/or uniqueness of local/global solutions to (4) when the data is no longer in  $L^\infty$ .
- The largest Lebesgue type space in which the equation is meaningful in the distributional sense is of Orlicz kind.
- **Ruf & Terraneo [2002]:** Local existence for small data in Orlicz space.

# Main results

## Good $H^1$ - Theory

### Theorem

Let  $u_0 \in H^1(\mathbb{R}^2)$ .

- 1 There exists a unique solution  $u$  to (4) in  $C([0, T], H^1)$ .
- 2 If  $f(u) = -u(e^{u^2} - 1)$ , then the (above) solution is global.
- 3 If  $f(u) = u(e^{u^2} - 1)$ ,  $u_0 \neq 0$  and  $J(u_0) \leq 0$ , then the (above) solution blows up in finite time.

Here the energy  $J$  is given by

$$J(u(t)) \stackrel{\text{def}}{=} \frac{1}{2} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}^2 - \int_{\mathbb{R}^2} F(u(t)) \, dx,$$

where

$$F(u) = \int_0^u f(v) \, dv.$$

## Nonuniqueness

### Theorem

Let  $B_1$  be the unit ball of  $\mathbb{R}^2$ . There exists infinitely many  $u_0 \in \mathcal{L}(B_1)$  such that the Cauchy problem

$$\left\{ \begin{array}{l} u_t - \Delta u = u(e^{u^2} - 1) \quad \text{in } B_1 \\ u(0) = u_0 \quad \text{in } B_1 \\ u|_{\partial B_1} = 0 \quad \text{for } t > 0 \end{array} \right. \quad (5)$$

has at least two solutions.

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# Background material

- $\|e^{t\Delta}\varphi\|_{L^\gamma} \leq C t^{\frac{1}{\gamma}-\frac{1}{\beta}} \|\varphi\|_{L^\beta}, t > 0, 1 \leq \beta \leq \gamma \leq \infty.$

- Using Duhamel's formula for the equation  $u_t - \Delta u = F(t, x)$ , we deduce

$$\|u\|_{L_T^\infty(H^1)} \leq C \left( \|u(0)\|_{H^1} + \|F\|_{L_T^1(H^1)} \right)$$

$$\|u\|_{L_T^\infty(H^1)} \leq C \left( \|u(0)\|_{H^1} + \|F\|_{L_T^1(L^2)} + T^{1/2} \|\nabla F\|_{L_T^\infty(L^1)} \right)$$

$$\|u\|_{L_T^\infty(L^\infty)} \leq C \left( \|u(0)\|_{L^\infty} + \|F\|_{L_T^1(L^\infty)} \right)$$

In order to control the nonlinear part in  $L_t^1(H_x^1)$ , we will use the following Moser-Trudinger type inequalities.

- $\int_{\mathbb{R}^2} \left( e^{\alpha|u(x)|^2} - 1 \right) dx \leq C_\alpha \|u\|_{L^2(\mathbb{R}^2)}^2$  whenever  $0 \leq \alpha < 4\pi$  and  $u \in H^1$  satisfying  $\|\nabla u\|_{L^2} \leq 1$ . The above inequality is false for  $\alpha \geq 4\pi$ .

- If we require  $\|u\|_{H^1} \leq 1$  rather than  $\|\nabla u\|_{L^2} \leq 1$ , we obtain

$$\sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left( e^{4\pi|u(x)|^2} - 1 \right) dx < \infty.$$

The above property is false for  $\alpha > 4\pi$ .

## Definition

Let  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \rightarrow 0^+} \phi(s), \quad \lim_{s \rightarrow \infty} \phi(s) = \infty.$$

The Orlicz space  $L^\phi$  is defined via the Luxembourg norm

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi \left( \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

- $\phi(s) = s^p, 1 \leq p < \infty \implies L^\phi = L^p.$
- $\phi_\alpha(s) = e^{\alpha s^2} - 1 \implies L^{\phi_\alpha} = L^{\phi_1} = \mathcal{L}.$

We shall use the following properties of Orlicz spaces.

- If  $T : L^1 \rightarrow L^1$  with norm  $M_1$  and  $T : L^\infty \rightarrow L^\infty$  with norm  $M_\infty$ , then  $T : L^\phi \rightarrow L^\phi$  with norm

$$M \leq \kappa_{\phi,d} \sup(M_1, M_\infty),$$

where  $\kappa_{\phi,d}$  depends only on  $\phi$  and the dimension  $d$ .

- For any  $p \geq 2$ ,  $\mathcal{L}(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$  and we have

$$\|u\|_{L^p} \leq \left(\Gamma\left(\frac{p}{2} + 1\right)\right)^{\frac{1}{p}} \|u\|_{\mathcal{L}}.$$

Recall the following parabolic regularizing effect due to **Brezis-Cazenave** that we will use to obtain a locally (in time) bounded solution to (4) with singular data. Consider the following linear heat equation with potential

$$\begin{cases} u_t - \Delta u - a(t, x)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0, \end{cases} \quad (6)$$

where  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^2$ .

### Theorem (**Brezis-Cazenave**)

*Let  $0 < T < \infty$ ,  $\sigma > 1$ , and let  $a \in L^\infty([0, T]; L^\sigma)$ . Given  $u_0 \in L^r$ ,  $1 \leq r < \infty$ , there exists a unique solution  $u \in C([0, T]; L^r) \cap L^\infty_{loc}([0, T]; L^\infty)$  of equation (6).*

# Sketches of the proofs

## Local existence in $H^1$

- We decompose the initial data  $u_0 \in H^1$  as

$$u_0 = u_{0,N} + u_0^N \quad (N \text{ large})$$

$$= H^1 \cap L^\infty + \text{small in } H^1$$

- We solve the IVP with data  $u_{0,N}$  to obtain a local solution  $v$ .
- To recover a solution of our original problem we solve a perturbed equation satisfied by  $w := u - v$  with small data  $u_0^N$ .

## Unconditional uniqueness

Let  $u, v \in \mathcal{C}([0, T]; H^1)$  be two solutions to (4) with same data  $u_0$  and set  $w := u - v$ . Then

$$w(t) = \int_0^t e^{(t-s)\Delta} a(s)w(s) ds,$$

where the potential  $a$  is given by

$$a(t, x) := \begin{cases} \frac{f(u) - f(v)}{w} & \text{if } w \neq 0 \\ f'(u) & \text{if } w = 0 \end{cases}$$

Remark that  $w \in L^\infty(0, T; L^q)$  for any  $2 \leq q < \infty$ .

To prove that  $w \equiv 0$  on  $[0, T]$ , we use the following lemma which can be seen as an extension to dimension two of [Brezis-Cazenave's](#) results about dimension  $d \geq 3$ .

### Lemma

Let  $a \in C([0, T]; L^p(\mathbb{R}^2))$  and  $u \in L^\infty((0, T); L^q(\mathbb{R}^2))$  with  $2 \leq q < \infty, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} \neq 1$ , and such that

$$u(t) = \int_0^t e^{(t-s)\Delta} a(s) u(s) ds, \quad 0 \leq t \leq T.$$

Then  $u = 0$  on  $[0, T]$ .

*A crucial fact here is that the potential is continuous in time. We don't know how to extend this lemma to the case when the potential is only  $L^\infty$  in time.*

- The uniqueness follows once we prove that  $a \in \mathcal{C}([0, T]; L^p(\mathbb{R}^2))$  for some  $1 < p < \infty$ .

- This basically follows from the fact that

$$u \in \mathcal{C}([0, T]; H^1(\mathbb{R}^2)) \implies e^{u^2} - 1 \in \mathcal{C}([0, T]; L^1(\mathbb{R}^2)).$$

## Global existence

Here we assume that  $f(u) = -u \left( e^{u^2} - 1 \right)$ .

### Proposition

Assume that  $u_0 \in H^1(\mathbb{R}^2)$  and  $u \in C([0, T]; H^1(\mathbb{R}^2))$  solution to (4). Then

$$\|u(t)\|_{L^\infty} \leq \sqrt{2} \|u_0\|_{L^2}, \quad 0 < t < T.$$

We use the level set energy inequality for  $c_k := M(1 - 2^{-k})$ , where  $M > 0$  will be chosen later. This idea was used by [Caffarelli-Vasseur](#) (**Annals of Mathematics (2010)**) for the quasi-geostrophic equation.

- This leads to the following energy inequality for the level set function  $u_k \stackrel{\text{def}}{=} (u - c_k)_+$ :

$$\frac{d}{dt} \int_{\mathbb{R}^2} |u_k(t)|^2 dx + 2 \int_{\mathbb{R}^2} |\nabla u_k(t)|^2 dx \leq 0. \quad (7)$$

- Let  $t_0 > 0$ , and denote by  $T_k \stackrel{\text{def}}{=} t_0(1 - 2^{-k})$  and

$$U_k \stackrel{\text{def}}{=} \sup_{t \geq T_k} \left( \int_{\mathbb{R}^2} |u_k(t, x)|^2 dx \right) + 2 \int_{T_k}^{+\infty} \int_{\mathbb{R}^2} |\nabla u_k(t, x)|^2 dx dt.$$

- By integrating (7) in time, and using some interpolation estimates, we obtain finally

$$U_k \leq A C^{k-1} U_{k-1}^{\frac{\alpha}{2}},$$

where

$$A \stackrel{\text{def}}{=} 2^{\alpha/2+4} M^{-\alpha} t_0^{-1} \|u_0\|_{L^2}^2, \quad \text{and} \quad C \stackrel{\text{def}}{=} 2^{\alpha+1}.$$

- Taking  $\alpha > 2$  and  $M \stackrel{\text{def}}{=} 2^{\frac{\alpha^2+10\alpha-12}{2\alpha(\alpha-2)}} t_0^{-\frac{1}{\alpha}} \|u_0\|_{L^2}$ , we get

$$\lim_{k \rightarrow \infty} U_k = 0.$$

- The conclusion follows by letting  $\alpha \rightarrow \infty$ .

## Blowing-up solutions

**Claim:** *If  $u_0 \in H^1 \setminus \{0\}$  with  $J(u_0) \leq 0$ , then  $T^* < \infty$ .*

The proof is quite standard. Set  $y(t) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^t \|u(s)\|_{L^2}^2 ds$ .

Since  $(uf(u) - 2F(u)) \geq \varepsilon F(u)$  for some  $\varepsilon > 0$ , we find that

$$y(t)y''(t) \geq (1 + \eta)(y'(t))^2 \quad \text{for some } \eta = \eta(\varepsilon) > 0.$$

The fact that this ordinary differential inequality blows up in finite time ensures the claim.

## Nonuniqueness

The proof is done in four steps. Set  $\Omega \stackrel{\text{def}}{=} B(0, 1) \subset \mathbb{R}^2$ .

**Step 1:** We construct a singular solution  $Q$  to

$$\left\{ \begin{array}{ll} -\Delta Q = f(Q) & \text{in } \Omega, \\ Q = 0 & \text{on } \partial\Omega, \\ Q(0) = \infty, & \text{and } Q > 0. \end{array} \right.$$

We look for singular solutions which are radially symmetric. Letting  $|x| \stackrel{\text{def}}{=} r = e^{-t}$ ,  $y(t) = Q(e^{-t})$ , we obtain

$$\begin{cases} -y''(t) = e^{-2t}f(y(t)), & t \geq 0 \\ y(0) = 0, & y(\infty) = \infty. \end{cases}$$

To solve the above problem, we proceed as follows.

- For any  $\alpha > 0$ , we consider the Cauchy problem

$$(\mathcal{P}_\alpha) \begin{cases} -y''(t) = e^{-2t}f(y(t)), & t \geq 0 \\ y(0) = 0, & y'(0) = \alpha, \end{cases}$$

and the associated elliptic problem

$$(\mathcal{E}_\alpha) \begin{cases} -\Delta u_\alpha = f(u_\alpha), & 0 < r < 1 \\ u_\alpha(1) = 0, & u_\alpha(0) = y_\alpha(\infty). \end{cases}$$

- Elliptic regularity + Existence and uniqueness of classical radial solution  $\implies \exists! \alpha_0$  s.t.  $\lim_{t \rightarrow \infty} y_{\alpha_0}(t) = \ell \in (0, \infty)$ .

Set

$$T(\alpha) = \sup \left\{ s \geq 0; \quad y_\alpha > 0 \text{ on } (0, s) \right\}$$

$$I = \left\{ \alpha > 0; \quad T(\alpha) < \infty \right\}$$

$$J = \left\{ \alpha > 0; \quad \alpha \notin I \right\}$$

The end of the proof of existence consists on showing that  $J$  is a non trivial interval.

**Step 2:** We prove that

- $-\Delta Q = f(Q)$  in  $\mathcal{D}'(\Omega)$ .
- $Q \in \mathcal{L}(\Omega)$ .
- $\lim_{r \rightarrow 0} \|Q\|_{\mathcal{L}(|x| < r)} = 0$ .

The above classical solution in  $\Omega \setminus \{0\}$  is extendable to a distribution solution in  $\Omega$  thanks to the following technical lemma.

### Lemma

Let  $u \in C^2(\Omega \setminus \{0\})$ ,  $u \geq 0$ , such that  $-\Delta u = f(u)$  in  $\Omega \setminus \{0\}$ .

Then

- 1  $f(u) \in L^1(\Omega)$ .
- 2 If  $(-\log r)^\alpha u^q \in L^1(\Omega)$  for some  $\frac{\alpha}{q-1} > 0$ , then

$$\Delta u + f(u) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

This lemma can be seen as an extension to dimension two of **Ni-Sacks'** result.

**Step 3:** We take  $Q$  as an initial data

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \\ u(0) = Q, u(|x| = 1) = 0 \end{cases} \quad (8)$$

- We prove that (8) has a local solution  $u \in L^\infty(\mathcal{L}(\Omega))$ .
- Since the potential  $e^{u^2} - 1$  is better than  $L^1$ , we can apply **Brezis-Cazenave's** result about regularization effect. This leads to  $u \in L^\infty(L^\infty(\Omega))$ .

**Step 4:** Now we are able to conclude.

$$u_0 = Q \longrightarrow v(t, x) = Q(x)$$

$$u_0 = Q \longrightarrow u(t, x) \in L^\infty(L^\infty(\Omega))$$

As  $Q(0) = \infty$ , we find  $u \neq v$ .

# Concluding remarks

- 1 We believe that there is no existence in  $H^s$  for  $s < 1$ .  
(Work in progress)
  
- 2 The blow-up analysis can be refined for positive initial energy.  
(Work in progress)

# Thank You