New Fixed Point Theorems in Banach Algebras Under Weak Topology Features and Applications to Nonlinear Integral Equations

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Notations

We denote by

- $\mathcal{E}$: Banach algebra.
- S: nonempty closed convex subset of $\mathcal{E}$.
- $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ are three operators.
- $\rightharpoonup$: weak convergence.

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- $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ are three operators.
- $\rightharpoonup$: weak convergence.

Definition 1

An operator $A : \mathcal{E} \longrightarrow \mathcal{E}$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset \mathcal{E}$.

Definition 2

An operator $A : \mathcal{E} \longrightarrow \mathcal{E}$ is said to be weakly sequentially continuous on $\mathcal{E}$ if for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$.

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Definition 3

An operator $A : \mathcal{E} \longrightarrow \mathcal{E}$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_A : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|Ax - Ay\| \leq \phi_A(\|x - y\|).$$

for all $x, y \in \mathcal{E}$, with $\phi_A(0) = 0$. Sometimes we call the function ϕ_A a $\mathcal{D}$-function of A on $\mathcal{E}$. If $\phi_A(r) = kr$ for some $k > 0$, then A is called a Lipschitzian function on $\mathcal{E}$ with the Lipschitz constant k. Further if $k < 1$, then A is called a contraction on $\mathcal{E}$ with the contraction k.

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Our aim

The purpose of this presentation is to prove the existence results of the following equation:

$$x = AxBx + Cx \tag{1}$$

History

R. W. Legget,
On certain nonlinear integral equations.
J. Math. Anal. Appl., 57 (1977), no. 2, 462-468.

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R. W. Legget,
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He proves the existence results of the following equation

$$x = x_0 + xBx \tag{2}$$

Under the condition of the compactness of the operator B, and the boundedness of the subset S such that $\sup_{x \in S} ||Bx|| < 1$ and $x_0 + xBx \in S$ for each $x \in S$.

History

B. C. Dhage,
A fixed point theorem in Banach algebras involving three operators with applications.
J. Kyun. Math., 44 (2004), 145 - 155.

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A fixed point theorem in Banach algebras involving three operators with applications.
J. Kyun. Math., 44 (2004), 145 - 155.

Let S be a closed, convex and bounded subset of a Banach algebra $\mathcal{E}$ and assume that

(*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

(*ii*) B is completely continuous (continuous and totally bounded),

(*iii*) $\forall y \in S, \forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

History

Then Equation (1) has at least one solution in S as soon as $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$, where $M := ||B(S)|| = \sup_{x \in S} \|Bx\|$.

History

A. Ben Amar, A. Jeribi and M. Mnif,
Some fixed point theorems and application to biological model.
Nume. Func. Anal. Opti. J., 29 (2008), 13 - 30.

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Theorem 1

Let $N : S \longrightarrow S$ be a weakly sequentially continuous map. If $N(S)$ is relatively weakly compact, then N has a fixed point in S.

Theorem 2

Assume that

(*i*) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.

(*ii*) $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous.

(*iii*) $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact.

(*iv*) $\forall y \in S, \forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

Then Equation (1) has at least one solution in S.

Proof sketch

- From assumption (i), it follows that for each y in S, there is a unique $x_y \in \mathcal{E}$ such that

$$\left(\frac{I-C}{A}\right)x_y = By. \tag{3}$$

or, equivalently

$$Ax_y By + Cx_y = x_y. \tag{4}$$

- Since the hypothesis (iv) holds, then $x_y \in S$. Therefore, we can define

$$\begin{cases} \mathcal{N} : S & \longrightarrow S \\ \quad\quad y & \longrightarrow \mathcal{N}y = x_y = \left(\frac{I-C}{A}\right)^{-1} By. \end{cases}$$

- By using the hypotheses (*ii*), (*iii*) and Theorem 1, we conclude that $\mathcal{N}$ has a fixed point y in S. Hence, y verifies Equation (1) i.e.,

$$AyBy + Cy = y.$$

Proposition 1

Assume that

(*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

(*ii*) A is regular on $\mathcal{E}$, i.e., A maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,

(*iii*) B is a bounded function with bound M.

Proposition 1

Assume that

(i) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

(ii) A is regular on $\mathcal{E}$, i.e., A maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,

(iii) B is a bounded function with bound M.

Then $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$ as soon as $M\phi_A(r)+\phi_C(r)<r$, for $r>0$.

Proof sketch

- Let y be fixed in S and define the mapping

$$\begin{cases} \varphi_y : \mathcal{E} & \longrightarrow \mathcal{E} \\ \qquad x & \longrightarrow \varphi_y(x) = AxBy + Cx. \end{cases}$$

- Let $x_1, x_2 \in \mathcal{E}$, the use of the assumption (i) leads to

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

Now, an application of a fixed point theorem of Boyd and Wong yields that there is a unique element $x_y \in \mathcal{E}$ such that

$$\varphi_y(x_y) = x_y$$

Hence, x_y verifies Equation (4) and so, by virtue of the hypothesis (ii), x_y verifies Equation (3). Therefore, the mapping $\left(\frac{I-C}{A}\right)^{-1}$ is well defined on $B(S)$ and $\left(\frac{I-C}{A}\right)^{-1} By = x_y$ and the desired result is deduced.

Theorem 3

Assume that

- (*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,
- (*ii*) A is regular on $\mathcal{E}$,
- (*iii*) B is weakly sequentially continuous and $B(S)$ is relatively weakly compact,
- (*iv*) $\left(\frac{I-C}{A}\right)^{-1}$ is weakly sequentially continuous on $B(S)$,
- (*v*) $\forall y \in S, \forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

Then Equation (1) has at least one solution in S as soon as $M\phi_A(r) + \phi_C(r) < r$, for $r > 0$.

Proof sketch

- From Proposition 1, it follows $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$. In virtue of assumption (v), we obtain

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S.$$

- By composition, $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous on S.
- Since $B(S)$ is relatively weakly compact together with assumption (iv), we get $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. The result is concluded immediately from Theorem 2.

Definition 4

We will say that the Banach algebra $\mathcal{E}$ satisfies condition $(\mathcal{P})$ if

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We will say that the Banach algebra $\mathcal{E}$ satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P})\begin{cases} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } \mathcal{E} \text{ such that} \\ x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y\text{, then } x_n y_n \rightharpoonup xy. \end{cases}$$

Theorem 4

Assume that

(*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

(*ii*) A is regular on $\mathcal{E}$,

(*iii*) A, B and C are weakly sequentially continuous on S,

(*iv*) $B(S)$ is bounded with bound M,

(*v*) $\left(\frac{I-C}{A}\right)^{-1}$ is weakly compact on $B(S)$,

(*vi*) $\forall y \in S, \forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

Then Equation (1) has at least one solution in S as soon as $M\phi_A(r) + \phi_C(r) < r$, for $r > 0$.

Proof sketch

- Similarly to the proof of Theorem 3, we obtain $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$,

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S$$

and $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. In view of Theorem 2, it suffices to establish that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous on S.

Proof sketch

- Similarly to the proof of Theorem 3, we obtain $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$,

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S$$

and $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. In view of Theorem 2, it suffices to establish that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous on S. To see this, let $\{u_n\}$ in S such that $u_n \rightharpoonup u$. Now, define $\{v_n\}$ in S by

$$v_n = \left(\frac{I-C}{A}\right)^{-1} B u_n$$

Then, there is a renamed subsequence such that

$$v_n = \left(\frac{I - C}{A}\right)^{-1} Bu_n \rightharpoonup v.$$

Therefore, from assumption (*iii*) and in view of condition ($\mathcal{P}$), we deduce that v verifies the following equation

$$v = \left(\frac{I - C}{A}\right)^{-1} Bu.$$

Thus, the whole sequence $\{u_n\}$ verifies

$$\left(\frac{I - C}{A}\right)^{-1} Bu_n = v_n \rightharpoonup v.$$

Theorem 5

Assume that

(*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

(*ii*) A is regular on $\mathcal{E}$,

(*iii*) A, B and C are weakly sequentially continuous on S,

(*iv*) $A(S), B(S)$ and $C(S)$ are relatively weakly compact,

(*v*) $\forall y \in S$, $\forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

Theorem 5

Assume that

(*i*) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,

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(*iii*) A, B and C are weakly sequentially continuous on S,

(*iv*) $A(S), B(S)$ and $C(S)$ are relatively weakly compact,

(*v*) $\forall y \in S$, $\forall x \in \mathcal{E}$: $x = AxBy + Cx \Longrightarrow x \in S$.

Then Equation (1) has at least one solution in S as soon as $M\phi_A(r) + \phi_C(r) < r$, for $r > 0$.

Proof sketch

- In view of preceding Theorem 4, it is enough to prove

$$\left(\frac{I-C}{A}\right)^{-1}B(S)$$ is relatively weakly compact.

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- In view of preceding Theorem 4, it is enough to prove

$\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. To see this, let $\{u_n\}$ in S and define $\{v_n\}$ in S by

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n.$$

Hence,

$$v_n = Av_nBu_n + Cv_n.$$

Now, by hypothesis (iv) and together with condition $(\mathcal{P})$, we deduce the existence of w, x, y such that

$$v_n \rightharpoonup wx + y.$$

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Example 1

To illustrate Theorem 3, we consider the following equation in $\mathcal{E} = \mathcal{C}(J, X)$, with $J = [0, 1]$ and $(X, \|.\|)$ is a Banach algebra:

$$x(t) = a(t)x(t) + (T_1x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u\right], \tag{5}$$

where $0 < \lambda < 1$ and u is non vanishing fixed vector of X.

Assumptions

$(H1)$

$a : J \longrightarrow X$ is a continuous function,

$(H2)$

$\sigma : J \longrightarrow J$ is a continuous and nondecreasing function,

$(H3)$

$q : J \longrightarrow \mathbb{R}$ is a continuous function,

Assumptions

$(H4)$

The operator $T_1 : \mathcal{C}(J,X) \longrightarrow \mathcal{C}(J,X)$ is such that

(a) T_1 is Lipschitzian with a Lipschitzian constant α,

(b) T_1 is regular on $\mathcal{C}(J,X)$,

(c) $\left(\frac{I}{T_1}\right)^{-1}$ is well defined on $\mathcal{C}(J,X)$,

(d) $\left(\frac{I}{T_1}\right)^{-1}$ is weakly sequentially continuous on $\mathcal{C}(J,X)$.

Assumptions

$(H5)$

The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$,

$(H6)$

There exists $0 < r_0 \leq 1$ such that

(a) $|p(t, s, x, y)| \leq r_0 - \|q\|_\infty$ for each $t, s \in J$; $x, y \in X$ such that $||x|| \leq r_0$ and $||y|| \leq r_0$,

(b) $\|T_1 x\|_\infty \leq \left(1 - \frac{\|a\|_\infty}{r_0}\right)\frac{1}{\|u\|}$ for each $x \in \mathcal{C}(J, X)$,

(c) $\alpha r_0\|u\| + \|a\|_\infty < 1$.

Theorem 1

Under assumptions $(H1)$-$(H6)$, *Equation (5) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{C}(J, X)$.

Remark

We denote by:

$$S := \{x \in \mathcal{C}(J, X),\ \|x\|_\infty \leq r_0\}$$

and

$$(Bx)(t) = \left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u, \quad x \in S.$$

When X is infinite dimensional, the subset $\mathcal{A}_{r_0} = \{x \in X;\ \|x\| \leq r_0\}$ is not compact. So, the restriction of p on $J \times J \times \mathcal{A}_{r_0} \times \mathcal{A}_{r_0}$ is not uniformly continuous. Thus, we note that the operator B in Equation (5) is not necessarily continuous on S.

Example 2

To illustrate Theorem 5, we consider the following equation in $\mathcal{E} = \mathcal{C}(J, X)$, with $J = [0, 1]$ and $(X, \|.\|)$ is a Banach algebra satisfying condition $(\mathcal{P})$:

$$x(t) = a(t) + (T_2x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u\right], \quad (6)$$

where $0 < \lambda < 1$ and u is non vanishing fixed vector of X.

Assumptions

$(H1)$

$a : J \longrightarrow X$ is a continuous function,

$(H2)$

$\sigma : J \longrightarrow J$ is a continuous and nondecreasing function,

$(H3)$

$q : J \longrightarrow \mathbb{R}$ is a continuous function,

Assumptions

$(H4)$

The operator $T_2 : \mathcal{C}(J,X) \longrightarrow \mathcal{C}(J,X)$ is such that

(a) T_2 is Lipschitzian with a Lipschitzian constant α,

(b) T_2 is regular on $\mathcal{C}(J,X)$,

(c) T_2 is weakly sequentially continuous on $\mathcal{C}(J,X)$,

(d) T_2 is weakly compact.

Assumptions

$(H5)$

The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$,

$(H6)$

There exists $r_0 > 0$ such that

(a) $|p(t, s, x, y)| \leq r_0 - \|q\|_\infty$ for each $t, s \in J$; $x, y \in X$ such that $||x|| \leq r_0$ and $||y|| \leq r_0$,

(b) $\|T_2 x\|_\infty \leq \left(1 - \frac{\|a\|_\infty}{r_0}\right) \frac{1}{\|u\|}$ for each $x \in \mathcal{C}(J, X)$,

(c) $\alpha r_0 \|u\| < 1$.

Theorem 2

Under assumptions $(H1)$-$(H6)$, *Equation (6) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{C}(J, X)$.

Thanks